1 METRIC TENSOR and REIMANIA METRIC

Let P and Q are two points having co ordinates (x, y) and $(x+dx, y+dy)$

$$P(x, y) \xrightarrow[\quad ds \quad]{} Q(x+dx, y+dy)$$

Let ds denote the distance between the two points

If the two points are very close, then

$$ds^2 = (x+dx-x)^2 + (y+dy-y)^2$$

$$= dx^2 + dy^2.$$

In tensor analysis, we can write

$$(ds)^2 = (dx^1)^2 + (dx^2)^2.$$

In three dimensional space the co ordinates are $P(x, y, z)$ and $Q(x+dx, y+dy, z+dz)$

$$P(x, y, z) \xrightarrow[\quad ds \quad]{} Q(x+dx, y+dy, z+dz)$$

Distance between the two points is given as

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

In tensor analysis, we assign

$$x^1 = x, \quad x^2 = y \quad \text{and } x^3 = z.$$

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$= 1 \cdot dx^1 dx^1 + 0 \cdot dx^1 dx^2 + 0 \cdot dx^1 dx^3$$
$$+ 0 \cdot dx^2 dx^1 + 1 \, dx^2 dx^2 + 0 \cdot dx^2 dx^3$$
$$+ 0 \cdot dx^3 dx^1 + 0 \, dx^3 dx^2 + 1 \cdot dx^3 dx^3.$$

Let us assume,

$g_{11} = 1$ $\quad g_{12} = 0$ $\quad g_{13} = 0$

$g_{21} = 0$ $\quad g_{22} = 1$ $\quad g_{23} = 0$

$g_{31} = 0$ $\quad g_{32} = 0$ $\quad g_{33} = 1$

Hence

$$(ds)^2 = g_{11}\,dx^1 dx^1 + g_{12}\,dx^1 dx^2 + g_{13}\,dx^1 dx^3$$
$$+ g_{21}\,dx^2 dx^1 + g_{22}\,dx^2 dx^2 + g_{23}\,dx^2 dx^3$$
$$+ g_{31}\,dx^3 dx^1 + g_{32}\,dx^3 dx^2 + g_{33}\,dx^3 dx^3.$$

In abbreviated form

$$(ds)^2 = g_{ij}\,dx^i dx^j \qquad i = 1, 2, 3$$
$$j = 1, 2, 3$$

It is to be remembered that here distance is taken along the straight line, which is not necessary.

Reimen introduced the idea in curved N-dimensional space as

$$ds^2 = g_{ij}\,dx^i dx^j \qquad i = 1, 2, \dots N$$
$$j = 1, 2 \dots N$$

where the coefficient g_{ij} may be functions of x^i; with the only restriction that the determinant of the co-efficient matrix be non singular i.e. $g \equiv \det(g_{ij}) \neq 0$. Such a space is called Riemanian space. If the coefficients g_{ij} are all independent of x^i, the space becomes a Euclidean space.

2. The fundamental tensor g_{ij} is a covariant symmetric tensor of rank 2

<u>Proof</u>:

Fundamental tensor is defined as

$$ds^2 = g_{ij}\, dx^i dx^j \qquad i, j = 1, 2, \dots n$$

ds is invariant; So for $\bar{x}$ system

$$ds^2 = \bar{g}_{ij}\, d\bar{x}^i d\bar{x}^j \quad (i, j = 1, 2, \dots n)$$

$$\therefore g_{ij}\, dx^i dx^j = \bar{g}_{ij}\, d\bar{x}^i d\bar{x}^j \quad \text{——— (1)}$$

Now,

$$\bar{x}^i = \bar{x}^i(x^1, x^2, \dots x^n)$$

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^1} dx^1 + \frac{\partial \bar{x}^i}{\partial x^2} dx^2 + \dots + \frac{\partial \bar{x}^i}{\partial x^n} dx^n$$

$$= \frac{\partial \bar{x}^i}{\partial x^\alpha} dx^\alpha \quad \text{——— (2)}$$

And $$d\bar{x}^j = \frac{\partial \bar{x}^j}{\partial x^\beta} dx^\beta \quad \text{——— (3)}$$

Putting the values of $d\bar{x}^i$ and $d\bar{x}^j$ from eqn (2) and eqn (3) in eqn (1), we get —

$$g_{ij}\, dx^i dx^j = \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial x^\alpha} dx^\alpha \frac{\partial \bar{x}^j}{\partial x^\beta} dx^\beta$$

$$= \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial \bar{x}^j}{\partial x^\beta} dx^\alpha dx^\beta .$$

Dummy suffix can be replaced by other suffixes also.

$$g_{\alpha\beta}\, dx^\alpha dx^\beta = \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial \bar{x}^j}{\partial x^\beta} dx^\alpha dx^\beta$$

$$\left(\bar{g}_{ij}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^j}{\partial x^\beta} - g_{\alpha\beta}\right) dx^\alpha dx^\beta = 0$$

As $dx^\alpha dx^\beta$ is arbitrary.

So, we may write as

$$\bar{g}_{ij}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^j}{\partial x^\beta} - g_{\alpha\beta} = 0$$

or $$g_{\alpha\beta} = \bar{g}_{ij}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^j}{\partial x^\beta}$$

or $$\boxed{\bar{g}_{ij} = \frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial x^\beta}{\partial \bar{x}^j} g_{\alpha\beta}} \text{ ——— } (4)$$

Here we see that the two partial derivatives denote that they are covariant tensors.

So, we find that g_{ij} is a covariant tensor of rank 2,

Now, we shall prove that fundamental tensor g_{ij} is a symmetric tensor.

We assume that

$$g_{ij} = \tfrac{1}{2}(g_{ij} + g_{ji}) + \tfrac{1}{2}(g_{ij} - g_{ji})$$

$$= B_{ij} + C_{ij} \text{ ——— } (5)$$

where, $B_{ij} = \frac{1}{2}(g_{ij} + g_{ji})$

and $C_{ij} = \frac{1}{2}(g_{ij} - g_{ji})$

Interchanging the suffixes of B_{ij}

$$B_{ji} = \frac{1}{2}(g_{ji} + g_{ij})$$
$$= \frac{1}{2}(g_{ij} + g_{ji})$$
$$= B_{ij}$$

So, B_{ij} is a symmetric tensor.

Again $C_{ij} = \frac{1}{2}(g_{ij} - g_{ji})$

Interchanging the dummy suffixes, we get

$$C_{ji} = \frac{1}{2}(g_{ji} - g_{ij}) = -\frac{1}{2}(g_{ij} - g_{ji})$$
$$= -C_{ij}$$

So, C_{ij} is a skew symmetric.

Using equation (5) into the expression of fundamental tensor we get.

$$g_{ij}\, dx^i dx^j = (B_{ij} + C_{ij})\, dx^i dx^j$$

or $(g_{ij} - B_{ij})\, dx^i dx^j = C_{ij}\, dx^i dx^j \longrightarrow (6)$

Interchanging the dummy suffixes in the R.H.S of eqn (6) we obtain

$$C_{ij}\, dx^i dx^j = C_{ji}\, dx^j dx^i$$
$$= -C_{ij}\, dx^i dx^j$$

$$2\, C_{ij}\, dx^i\, dx^j = 0$$

$$\Rightarrow \quad C_{ij}\, dx^i\, dx^j = 0 \qquad \text{------ (7)}$$

From eq^n (6) and eq^n (7)

$$(g_{ij} - B_{ij})\, dx^i\, dx^j = 0$$

Since $dx^i\, dx^j$ is arbitrary

$$g_{ij} - B_{ij} = 0$$

$$\Rightarrow \quad g_{ij} = B_{ij} = \text{A symmetric tensor}$$

So, we may conclude that

g_{ij} is a covariant symmetric tensor of rank 2.

• 3 • Show that $g_{ij}\, dx^i\, dx^j$ is an invariant.

Since g_{ij} is a covariant vector

$$\bar{g}_{ij}\, d\bar{x}^i\, d\bar{x}^j = \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^\beta}{\partial \bar{x}^j}\, g_{\alpha\beta}\, d\bar{x}^i\, d\bar{x}^j \qquad \text{------} (1)$$

Now, $$x^\alpha = x^\alpha(\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n)$$

$$\therefore \quad dx^\alpha = \frac{\partial x^\alpha}{\partial \bar{x}^i}\, d\bar{x}^i \qquad \text{------} (2)$$

Similarly, we can write

$$dx^\beta = \frac{\partial x^\beta}{\partial \bar{x}^j}\, d\bar{x}^j \qquad \text{------} (3)$$

Substituting the results of eqn (2) and eqn (3) in eqn (1)

$$\bar{g}_{ij}\, d\bar{x}^i\, d\bar{x}^j = \left(\frac{\partial x^\alpha}{\partial \bar{x}^i}\, d\bar{x}^i\right)\left(\frac{\partial x^\beta}{\partial \bar{x}^j}\, d\bar{x}^j\right) g_{\alpha\beta}$$

$$= dx^\alpha\, dx^\beta\, g_{\alpha\beta}.$$

Rearranging, we get –

$$\bar{g}_{ij}\, d\bar{x}^i\, d\bar{x}^j = g_{\alpha\beta}\, dx^\alpha\, dx^\beta.$$

Substituting dummy suffix α and β by i and j we get

$$\bar{g}_{ij}\, d\bar{x}^i\, d\bar{x}^j = g_{ij}\, dx^i\, dx^j$$

In both system of coordinates Reimanian matrix is same i.e. invariant.

4. Conjugate metric tensor (Contravariant tensor).

First we define conjugate (or reciprocal) symmetric tensor.

We consider a covariant symmetric tensor A_{ij} of rank two. Let d denote the determinant $|A_{ij}|$ with the element A_{ij}

i.e. $d = |A_{ij}|$ and $d \neq 0$

Now, conjugate (Reciprocal tensor) tensor of A_{ij} is defined as

$$A^{ij} = \frac{\text{cofactor of } A_{ij}}{d} = \frac{B_{ij}}{d}.$$

<u>Theorem</u>

If B_{ij} is the cofactor of A_{ij} having the determinant $d = |A_{ij}| \neq 0$ and conjugate tensor $A^{ij} = \frac{B_{ij}}{d}$

then prove that $A_{ij} A^{kj} = \delta^k_i$

From the properties of the determinant, we have two results

i) $A_{ij} B_{ij} = d \Rightarrow A_{ij} \frac{B_{ij}}{d} = 1$

$$\Rightarrow A_{ij} A^{ij} = 1 \longrightarrow (1)$$

Given $A^{ij} = \frac{B_{ij}}{d}$

ii) $A_{ij} B_{kj} = 0$

$\Rightarrow A_{ij} \dfrac{B_{kj}}{d} = 0$ if $d \neq 0$

$\Rightarrow A_{ij} A^{kj} = 0$ if $i \neq k$ ——— (2)

From the relations (1) and (2)

$$A_{ij} A^{kj} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$\therefore A_{ij} A^{kj} = \delta_i^k$ ——— (3)

- Conjugate metric tensor

The conjugate metric tensor to g_{ij} is defined by

$$g^{ij} = \frac{A^{ij}}{g}, \quad \text{where } \det(g_{ij})_{n\times n} = g$$

and A_{ij} = cofactor of (g_{ij})

From the theorem proved above

$$g_{ij} g^{kj} = \delta_i^k$$

If we take $k = i$

$$g_{ij} g^{ij} = \delta_i^i = n.$$

Again $|g^{ij}| = \left|\dfrac{A^{ij}}{g}\right|_{n\times n} = \dfrac{1}{g^n} |A_{ij}|$

$$= \frac{g^{n-1}}{g^n} = \frac{1}{g}.$$

5. Associate Tensor

A covariant vector A_i is said to be associated to A^j if

$$A_i = g_{ij} A^j .$$

Theorem

If $A_i = g_{ij} A^j$ then $A^j = g^{ij} A_i$

Proof

Given $A_i = g_{ij} A^j$

Multiplying the given expression with g^{il}

$$g^{il} A_i = g^{il} g_{ij} A^j$$

$$\Rightarrow \quad g^{il} A_i = \delta^l_j A^j = A^l .$$

$$\Rightarrow \quad A^l = g^{il} A_i$$

Putting $l = j$.

$$A^j = g^{ij} A_i .$$

6. Magnitude of a vector

Let the magnitude or length A of the covariant vector A_i then, A is define by $A = \sqrt{g^{ij} A_i A_j}$

$$\Rightarrow \quad A^2 = g^{ij} A_i A_j = (g^{ij} A_i) A_j = A^j A_j \quad \text{---(1)}$$

Similarly we can write

$$g_{ij} A^i A^j = A_j A^j = A^j A_j \qquad (2)$$

(By commutativity)

From eq^n (1) and eq^n (2), we write

$$g^{ij} A_i A_j = g_{ij} A^i A^j$$

or

$$A^2 = g_{ij} A^i A^j \qquad (3)$$

From eq. (1), eq. (2) and eq^n (3) we have

$$g^{ij} A_i A_j = A^j A_j = A_j A^j = g_{ij} A^i A^j .$$

• 7 • Angle between the two vectors

Let A^i and B^i are two vectors and the angle between the two vectors is θ; then

$$\cos\theta = \frac{g_{ij} A^i B^j}{\sqrt{g_{ij} A^i A^j}\sqrt{g_{pq} g^p g^q}}$$

Two vectors A^i and B^i are said to be orthogonal to one another if the angle between them is a right angle. The necessary and sufficient condition for the orthogonality of the two vectors (A^i) and (B^i) is that

$$g_{ij} A^i B^j = 0$$

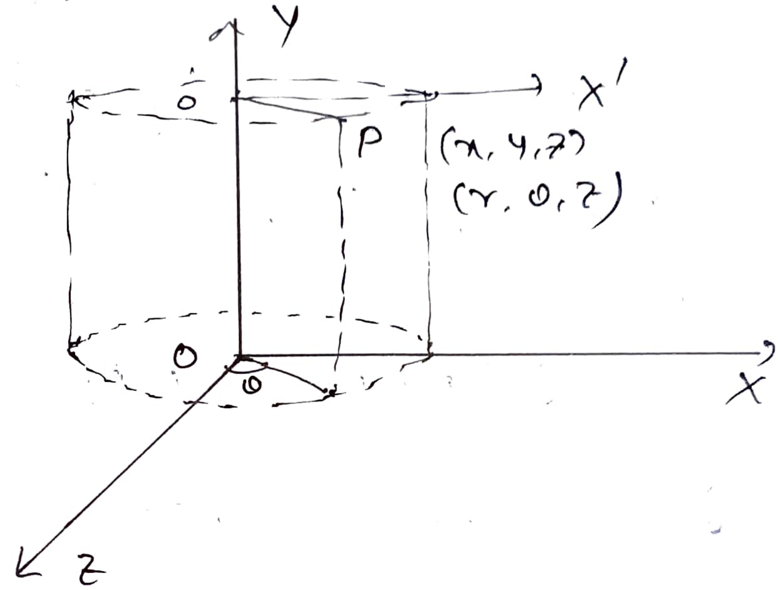
- Line element or metric in cylindrical system and its conjugate tensor :

- Let (x, y, z) be the Cartesian coordinates and (r, θ, z) be the cylindrical co-ordinates of a point. The co-ordinates are related by the following equations.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$



we find that

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial x}{\partial z} = 0$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta \qquad \frac{\partial y}{\partial z} = 0$$

$$\frac{\partial z}{\partial r} = 0 \qquad \frac{\partial z}{\partial \theta} = 0 \qquad \frac{\partial z}{\partial z} = 1.$$

Further, we consider

$$x = x^1, \qquad y = x^2, \qquad z = x^3$$

Its corresponding values in cylindrical system of co-ordinates are

$$r = \bar{x}^1, \qquad \theta = \bar{x}^2 \quad \text{and} \quad z = \bar{x}^3.$$

Now, we write line element in Cartesian coordinate

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= 1.\,dx\,dx + 0.\,dx\,dy + 0\,dx\,dz$$
$$+ 0.\,dy\,dx + 1.\,dy\,dy + 0.\,dy\,dz$$
$$+ 0.\,dz\,dx + 0.\,dz\,dy + 1.\,dz\,dz$$

$\therefore$ $g_{11} = 1$ $\quad g_{12} = 0$ $\quad g_{13} = 0$

$g_{21} = 0$ $\quad g_{22} = 1$ $\quad g_{23} = 0$

$g_{31} = 0$ $\quad g_{32} = 0$ $\quad g_{33} = 1$.

Hence we obtain the metric tensor in Cartesian coordinates as

$$(g_{ij})_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we try to find metric in cylindrical system of coordinates

We first consider metric in cylindrical coordinate as $\bar{g}_{ij}$ while the metric in cartesian coordinate is expressed as g_{ij}.

Now, we apply transformation rule to have.

$$\bar{g}_{ij} = \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^\beta}{\partial \bar{x}^j} g_{\alpha\beta}$$

From the expression of metric tensor in Cartesian coordinates, we see that

$$g_{ij} = 0 \quad \text{when } i \neq j$$

According to the theory of tensor if a tensor vanishes in one system, then its corresponding component vanishes in every system. In this case cylindrical components vanish.

i.e. $\bar{g}_{ij} = 0$

So, there will be only three non-zero terms. Let us find those terms expanding the transformation relation.

$$\bar{g}_{ij} = \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^\beta}{\partial \bar{x}^j} g_{\alpha\beta}$$

$$= \frac{\partial x^1}{\partial \bar{x}^i} \frac{\partial x^1}{\partial \bar{x}^j} g_{11} + \frac{\partial x^2}{\partial \bar{x}^i} \frac{\partial x^2}{\partial \bar{x}^j} g_{22} + \frac{\partial x^3}{\partial \bar{x}^i} \frac{\partial x^3}{\partial \bar{x}^j} g_{33}$$

Now, putting $j = i$

$$\bar{g}_{11} = \frac{\partial x^1}{\partial \bar{x}^1} \frac{\partial x^1}{\partial \bar{x}^1} g_{11} + \frac{\partial x^2}{\partial \bar{x}^1} \frac{\partial x^2}{\partial \bar{x}^1} g_{22} + \frac{\partial x^3}{\partial \bar{x}^1} \frac{\partial x^3}{\partial \bar{x}^1} g_{33}$$

$$= \frac{\partial x}{\partial r} \frac{\partial x}{\partial r} \times 1 + \frac{\partial y}{\partial r} \frac{\partial y}{\partial r} \times 1 + \frac{\partial z}{\partial r} \frac{\partial z}{\partial r} \times 1$$

$$= \cos^2\theta + \sin^2\theta + 0$$

$$= 1.$$

$$\bar{g}_{22} = \frac{\partial x^1}{\partial \bar{x}^2}\frac{\partial x^1}{\partial \bar{x}^2} g_{11} + \frac{\partial x^2}{\partial \bar{x}^2}\frac{\partial x^2}{\partial \bar{x}^2} g_{22} + \frac{\partial x^3}{\partial \bar{x}^2}\frac{\partial x^3}{\partial \bar{x}^2} g_{33}$$

$$= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} \times 1 + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} \times 1 + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} \times 1$$

$$= r^2 \sin^2\theta + r^2 \cos^2\theta + 0$$

$$= r^2$$

$$\bar{g}_{33} = \frac{\partial x^1}{\partial \bar{x}^3}\frac{\partial x^1}{\partial \bar{x}^3} g_{11} + \frac{\partial x^2}{\partial \bar{x}^3}\frac{\partial x^2}{\partial \bar{x}^3} g_{22} + \frac{\partial x^3}{\partial \bar{x}^3}\frac{\partial x^3}{\partial \bar{x}^3} g_{33}$$

$$= \frac{\partial x}{\partial z}\frac{\partial x}{\partial z} \times 1 + \frac{\partial y}{\partial z}\frac{\partial y}{\partial z} \times 1 + \frac{\partial z}{\partial z}\frac{\partial z}{\partial z} \times 1$$

$$= 0 + 0 + 1$$

$$= 1.$$

Hence, metric tensor in cylindrical system is given as

$$[\bar{g}_{ij}]_{3\times 3} = \begin{bmatrix} \bar{g}_{11} & \bar{g}_{12} & \bar{g}_{13} \\ \bar{g}_{21} & \bar{g}_{22} & \bar{g}_{23} \\ \bar{g}_{31} & \bar{g}_{32} & \bar{g}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we construct the line element in cylindrical co-ordinates as

$$ds^2 = \bar{g}_{ij}\, d\bar{x}^i\, d\bar{x}^j$$

$$= \bar{g}_{11}\, d\bar{x}^1 d\bar{x}^1 + \bar{g}_{22}\, d\bar{x}^2 d\bar{x}^2 + \bar{g}_{33}\, d\bar{x}^3 d\bar{x}^3$$

$$= 1 \cdot dr\, dr + r^2\, d\theta\, d\theta + 1 \cdot dz\, dz$$

$$= dr^2 + r^2 d\theta^2 + dz^2.$$

Now we find the conjugate tensor of $\bar{g}_{ij}$.

From the definition of conjugate tensor

$$\bar{g}^{ij} = \frac{\text{cofactor of } \bar{g}_{ij}}{|\bar{g}_{ij}|}$$

Now, $\bar{g}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore |\bar{g}_{ij}| = r^2 = g'$ (Let)

Now cofactor of $\bar{g}_{11} = \bar{A}_{11} = r^2$

" of $\bar{g}_{22} = \bar{A}_{22} = 1$

" of $\bar{g}_{33} = \bar{A}_{33} = r^2$

$\therefore \bar{g}^{11} = \frac{\bar{A}_{11}}{g'} = \frac{r^2}{r^2} = 1$

$\bar{g}^{22} = \frac{\bar{A}_{22}}{g'} = \frac{1}{r^2}$.

$\bar{g}^{33} = \frac{\bar{A}_{33}}{g'} = \frac{r^2}{r^2} = 1$.

Again, $\bar{g}^{ij} = \frac{\bar{A}^{ij}}{g'}$, for $i \neq j$ $\quad \bar{A}^{ij} = 0$

$= \frac{0}{g'} = 0$

So, we may write conjugate tensor as

$$(\bar{g}^{ij})_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Find the matrix and component of first and second fundametal tensors in spherical co-ordinates.

- Solution

Let (x^1, x^2, x^3) be the cartesian coordinates and $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ be the spherical coordinates of a point. The spherical coordinates are given by

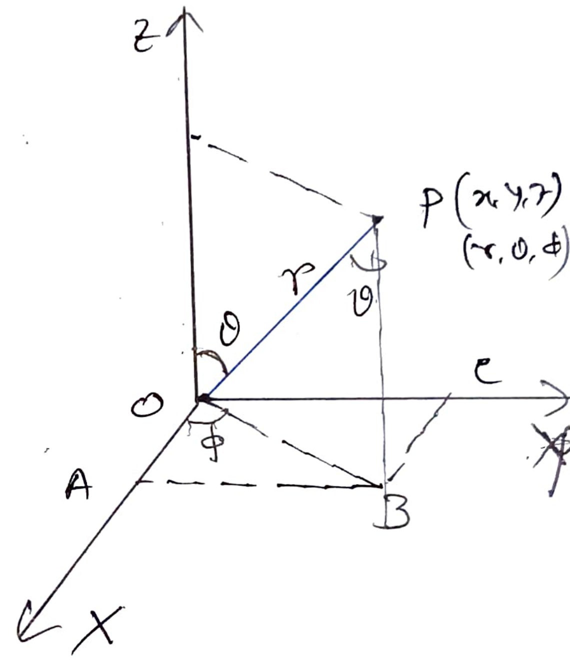
$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

so that $x^1 = x,\ x^2 = y,\ x^3 = z$

and $\bar{x}^1 = r,\ \bar{x}^2 = \theta,\ \bar{x}^3 = \phi$



Now we evaluate

$$\frac{\partial x}{\partial r} = \sin\theta \cos\phi \qquad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta \sin\phi, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta, \quad \frac{\partial z}{\partial \phi} = 0$$

Now, we have

$$g_{11} = g_{22} = g_{33} = 1$$

and $g_{ij} = 0$ for $i \neq j$.

On transformation,

$$g_{ij} = \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^\beta}{\partial \bar{x}^j} g_{\alpha\beta}$$

(Since g_{ij} is covariant tensor of rank two, where, $i, j = 1, 2, 3$)

$$\bar{g}_{ij} = g_{11} \frac{\partial x^1}{\partial \bar{x}^i} \frac{\partial x^1}{\partial \bar{x}^j} + g_{22} \frac{\partial x^2}{\partial \bar{x}^i} \frac{\partial x^2}{\partial \bar{x}^j} + g_{33} \frac{\partial x^3}{\partial \bar{x}^i} \frac{\partial x^3}{\partial \bar{x}^j} \longrightarrow (1)$$

Putting $i = j = 1$

$$\bar{g}_{11} = g_{11} \frac{\partial x^1}{\partial \bar{x}^1} \frac{\partial x^1}{\partial \bar{x}^1} + g_{22} \frac{\partial x^2}{\partial \bar{x}^1} \frac{\partial x^2}{\partial \bar{x}^1} + g_{33} \frac{\partial x^3}{\partial \bar{x}^1} \frac{\partial x^3}{\partial \bar{x}^1}$$

$$= 1 \times \frac{\partial x}{\partial r} \frac{\partial x}{\partial r} + 1 \times \frac{\partial y}{\partial r} \frac{\partial y}{\partial r} + 1 \times \frac{\partial z}{\partial r} \frac{\partial z}{\partial r}$$

$$= \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta$$

$$= \sin^2\theta (\cos^2\phi + \sin^2\phi) + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta = 1.$$

Again putting $i = j = 2$ in the expression (1)

$$\bar{g}_{22} = \frac{\partial x^1}{\partial \bar{x}^2} \frac{\partial x^1}{\partial \bar{x}^2} g_{11} + g_{22} \frac{\partial x^2}{\partial \bar{x}^2} \frac{\partial x^2}{\partial \bar{x}^2} + g_{33} \frac{\partial x^3}{\partial \bar{x}^2} \frac{\partial x^3}{\partial \bar{x}^2}$$

$$= \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \theta}$$

$$= r^2 \cos^2\theta \cos^2\phi + r^2 \cos^2\theta \sin^2\phi + r^2 \sin^2\theta$$

$$= r^2 \cos^2\theta (\cos^2\phi + \sin^2\phi) + r^2 \sin^2\theta$$

$$= r^2 \cos^2\theta + r^2 \sin^2\theta$$

$$= r^2$$

Again putting $i = j = 2$ in equation - 1

$$\bar{g}_{33} = \frac{\partial x^1}{\partial \bar{x}^3}\frac{\partial x^1}{\partial \bar{x}^3} g_{11} + g_{22}\frac{\partial x^2}{\partial \bar{x}^3}\frac{\partial x^2}{\partial \bar{x}^3} + g_{33}\frac{\partial x^3}{\partial \bar{x}^3}\frac{\partial x^3}{\partial \bar{x}^3}$$

$$= \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \phi}$$

$$= r^2 \sin^2\theta \sin^2\phi + r^2 \sin^2\theta \cos^2\phi + 0$$

$$= r^2 \sin^2\theta (\sin^2\phi + \cos^2\phi)$$

$$= r^2 \sin^2\theta.$$

$\therefore \quad \bar{g}_{11} = 1, \quad \bar{g}_{22} = r^2$ and $\bar{g}_{33} = r^2 \sin^2\theta$.

$\bar{g}_{ij} = 0$ for $i \neq j$.

The metric tensor or first fundamental tensor is

$$\bar{g}_{ij} = \begin{bmatrix} \bar{g}_{11} & \bar{g}_{12} & \bar{g}_{13} \\ \bar{g}_{21} & \bar{g}_{22} & \bar{g}_{23} \\ \bar{g}_{31} & \bar{g}_{32} & \bar{g}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

$$g = |\bar{g}_{ij}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = r^4 \sin^2\theta.$$

The metric in spherical co-ordinates is

$$ds^2 = \bar{g}_{ij}\, d\bar{x}^i d\bar{x}^j ; \quad i, j = 1, 2, 3$$

$$= \bar{g}_{11}\, d\bar{x}^1 d\bar{x}^1 + \bar{g}_{22}\, d\bar{x}^2 d\bar{x}^2 + \bar{g}_{33}\, d\bar{x}^3 d\bar{x}^3$$

$$= (dr)^2 + r^2 (d\theta)^2 + (r \sin\theta\, d\phi)^2$$

$$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

The second fundemental tensor or conjugate tensor is $\bar{g}^{ij} = \frac{\bar{B}_{ij}}{g'}$

where $\bar{B}_{ij}$ is cofactor of $\bar{g}_{ij}$

g' is determinant of $\bar{g}_{ij} = r^4 \sin^2\theta$

Now, cofactor of $\bar{g}_{11}$ is $\bar{B}_{11} = r^4 \sin^2\theta$

" " $\bar{g}_{22}$ is $\bar{B}_{22} = r^2 \sin^2\theta$

" " $\bar{g}_{33}$ is $\bar{B}_{33} = r^2$

$$\therefore \quad \bar{g}^{11} = \frac{\bar{B}_{11}}{g'} = \frac{r^4 \sin^2\theta}{r^4 \sin^2\theta} = 1$$

$$\bar{g}^{22} = \frac{\bar{B}_{22}}{g'} = \frac{r^2 \sin^2\theta}{r^4 \sin^2\theta} = \frac{1}{r^2}$$

$$\bar{g}^{33} = \frac{\bar{B}_{33}}{g'} = \frac{r^2}{r^4 \sin^2\theta} = \frac{1}{r^2 \sin^2\theta}$$

and $\bar{g}^{12} = \bar{g}^{13} = \bar{g}^{21} = \bar{g}^{23} = \bar{g}^{31} = \bar{g}^{32} = 0$

Hence, the fundamental tensor in matrix form is

$$(\bar{g}^{ij})_{3\times 3} = \begin{bmatrix} \bar{g}^{11} & \bar{g}^{12} & \bar{g}^{13} \\ \bar{g}^{21} & \bar{g}^{22} & \bar{g}^{23} \\ \bar{g}^{31} & \bar{g}^{32} & \bar{g}^{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix}$$

Problem

If the metric is given by

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6\,dx^1 dx^2 + 4\,dx^2 dx^3$$

Evaluate (i) g and (ii) g^{ij}.

Solution

The metric is

$$ds^2 = g_{ij}\,dx^i dx^j \quad (i, j = 1, 2, 3)$$

$$\Rightarrow ds^2 = g_{11}(dx^1)^2 + g_{12}\,dx^1 dx^2 + g_{13}\,dx^1 dx^3 + g_{21}\,dx^1 dx^2 + g_{22}\,dx^2 dx^2 + g_{23}\,dx^2 dx^3 + g_{31}\,dx^3 dx^1 + g_{32}\,dx^3 dx^2 + g_{33}\,dx^3 dx^3$$

Since $g_{ij} = g_{ji}$ (g_{ij} is symmetric)

$$\Rightarrow g_{12} = g_{21}, \quad g_{23} = g_{32}, \quad g_{13} = g_{31}$$

So,

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{12}\,dx^1 dx^2 + 2g_{23}\,dx^2 dx^3 + 2g_{13}\,dx^1 dx^3 \quad (1)$$

Now, the given metric

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6\,dx^1 dx^2 + 4\,dx^2 dx^3 \quad (2)$$

Comparing equation (1) and eqn (2), we have

$g_{11} = 5$ $\quad g_{22} = 3$ $\quad g_{33} = 4$

$2g_{12} = -6 \quad \Rightarrow \quad g_{12} = -3 = g_{21}$

$2g_{23} = 4 \quad \Rightarrow \quad g_{23} = 2 = g_{32}$

and $\quad g_{13} = 0 = g_{31}$.

So, the metric tensor

$$g_{ij} = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\therefore \; g = |g_{ij}| = 5(12-4) - 3\times(0+12)$$
$$= 40 - 36 = 4.$$

ii) Let B_{ij} be the cofactor of g_{ij} in g

$B_{11} =$ cofactor of $g_{11} = \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} = 12 - 4 = 8$

$B_{12} = \; \cdots \; g_{12} = -\begin{vmatrix} -3 & 2 \\ 0 & 4 \end{vmatrix} = 12 = B_{21}$

$B_{13} = \; \cdots \; g_{13} = \begin{vmatrix} -3 & 3 \\ 0 & 2 \end{vmatrix} = -6 = B_{31}$

$B_{23} = \; \cdots \; g_{23} = -\begin{vmatrix} 5 & -3 \\ 0 & 2 \end{vmatrix} = -10 = B_{32}$

$B_{22} = \; \cdots \; g_{22} = \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} = 20$

$B_{33} = \; \cdots \; g_{33} = \begin{vmatrix} 5 & -3 \\ -3 & 3 \end{vmatrix} = 15 - 9 = 6$

Conjugate metric

$$g^{ij} = \frac{B_{ij}}{g}$$

We have

$$g^{11} = \frac{B_{11}}{g} = \frac{8}{4} = 2$$

$$g^{22} = \frac{B_{22}}{g} = \frac{20}{4} = 5$$

$$g^{33} = \frac{B_{33}}{g} = \frac{6}{4} = \frac{3}{2}$$

$$g^{12} = g^{21} = \frac{B_{12}}{g} = \frac{12}{4} = 3$$

$$g^{13} = g^{31} = \frac{B_{13}}{g} = \frac{-6}{4} = \frac{-3}{2}$$

$$g^{23} = g^{32} = \frac{B_{23}}{g} = -\frac{10}{4} = -\frac{5}{2}$$

Hence conjugate metric tensor is

$$g^{ij} = \begin{bmatrix} 2 & 3 & -\frac{3}{2} \\ 3 & 5 & -5/2 \\ -\frac{3}{2} & -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$$