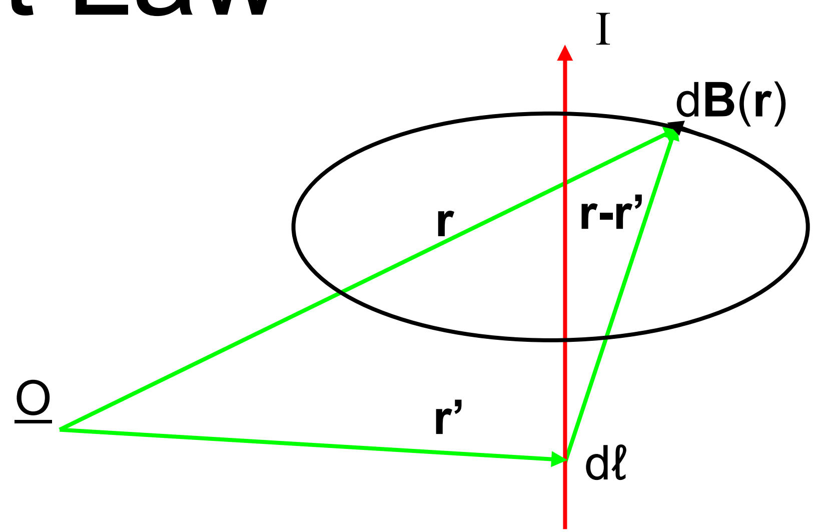
# Biot-Savart Law




- The analogue of Coulomb’s Law is the Biot-Savart Law
- Consider a current loop (I)
- For element d**ℓ** there is an associated element field d**B**

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_o I}{4\pi} \frac{d\ell \times (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3}$$

d**B** perpendicular to both d**ℓ** and **r** - **r’**

Inverse square dependence on distance

$\mu_o/4\pi = 10^{-7}$ Hm$^{-1}$

Integrate to get B-S Law

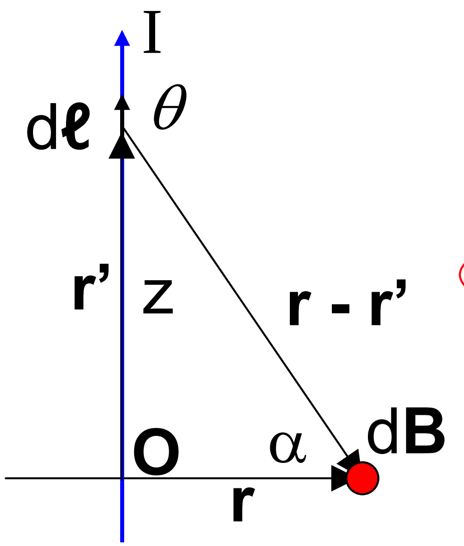
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o I}{4\pi} \oint_\ell \frac{d\ell \times (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3}$$

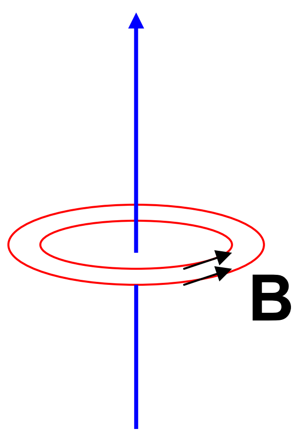
# Biot-Savart Law examples

(1) Infinite straight conductor

$d\ell$ and **r, r'** in the page
d**B** is out of the page

**B** forms concentric circles about the conductor




$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o I}{4\pi} \int_{-\infty}^{\infty} \frac{d\ell \times (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3}$$

$$d\ell \times (\mathbf{r} - \mathbf{r'}) = |d\ell||\mathbf{r} - \mathbf{r'}|\sin\theta\,\hat{n}$$

$$|\mathbf{r} - \mathbf{r'}|^2 = r^2 + z^2$$

$$\theta = \pi/2 + \alpha \qquad \sin\theta = \cos\alpha = \frac{r}{(r^2 + z^2)^{1/2}}$$

$$\frac{d\ell \times (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3} = \frac{|d\ell||\mathbf{r} - \mathbf{r'}|\sin\theta}{|\mathbf{r} - \mathbf{r'}|^3}\hat{n} = \frac{r\,dz}{(r^2 + z^2)^{3/2}}\hat{n}$$

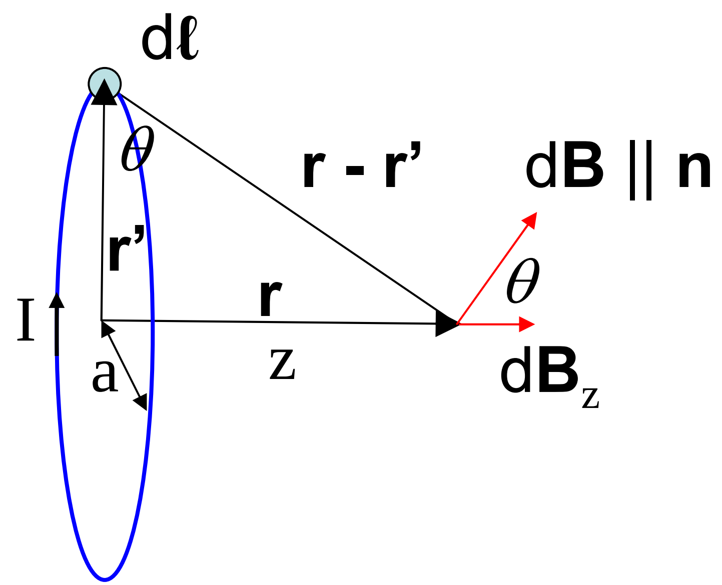
$$\mathbf{B} = \frac{\mu_o I}{4\pi} \int_{-\infty}^{\infty} \frac{r\,dz}{(r^2 + z^2)^{3/2}}\;\hat{n}$$

$$\int_{-\infty}^{\infty} \frac{r\,dz}{(r^2 + z^2)^{3/2}} = \frac{2}{r}$$

$$\boxed{\mathbf{B} = \frac{\mu_o I}{2\pi r}\hat{n}}$$

# Biot-Savart Law examples

(2) axial field of circular loop




Loop perpendicular to page, radius *a*

dℓ out of page at top and **r, r'** in the page
On-axis element d**B** is in the page,
perpendicular to **r** - **r'**, at $\theta$ to axis.

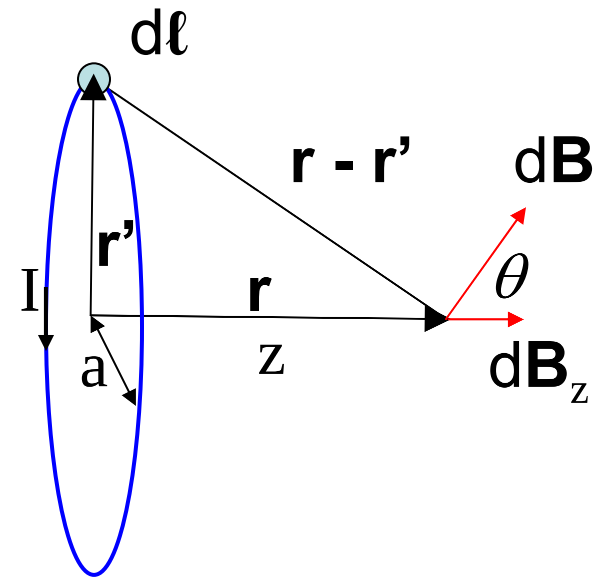
Magnitude of element d**B**

$$dB = \frac{\mu_o I}{4\pi}\frac{d\ell \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\mu_o I}{4\pi}\frac{|d\ell|}{|\mathbf{r}-\mathbf{r}'|^2}\hat{n} \Rightarrow dB_z = \frac{\mu_o I}{4\pi}\frac{d\ell}{|\mathbf{r}-\mathbf{r}'|^2}\cos\theta$$

$$\cos\theta = \frac{a}{|\mathbf{r}-\mathbf{r}'|} = \frac{a}{(a^2+z^2)^{1/2}}$$

Integrating around loop, only z-component of d**B** contributes net result

# On-axis field of circular loop




$$B_{on-axis} = \oint dB_z = \frac{\mu_o I}{4\pi |\mathbf{r} - \mathbf{r}'|^2} \cos\theta \oint d\ell$$

$$= \frac{\mu_o I}{4\pi |\mathbf{r} - \mathbf{r}'|^2} \cos\theta (2\pi a) = \frac{\mu_o I a^2}{2|\mathbf{r} - \mathbf{r}'|^3}$$

Introduce axial distance z, where $|\mathbf{r}-\mathbf{r}'|^2 = a^2 + z^2$

$$B_{on-axis} = \frac{\mu_o I a^2}{2(a^2 + z^2)^{3/2}}$$

2 limiting cases

$$B_{on-axis}^{z=0} = \frac{\mu_o I}{2a} \quad \text{and} \quad B_{on-axis}^{z>>a} \approx \frac{\mu_o I a^2}{2z^3}$$

# Magnetic dipole moment

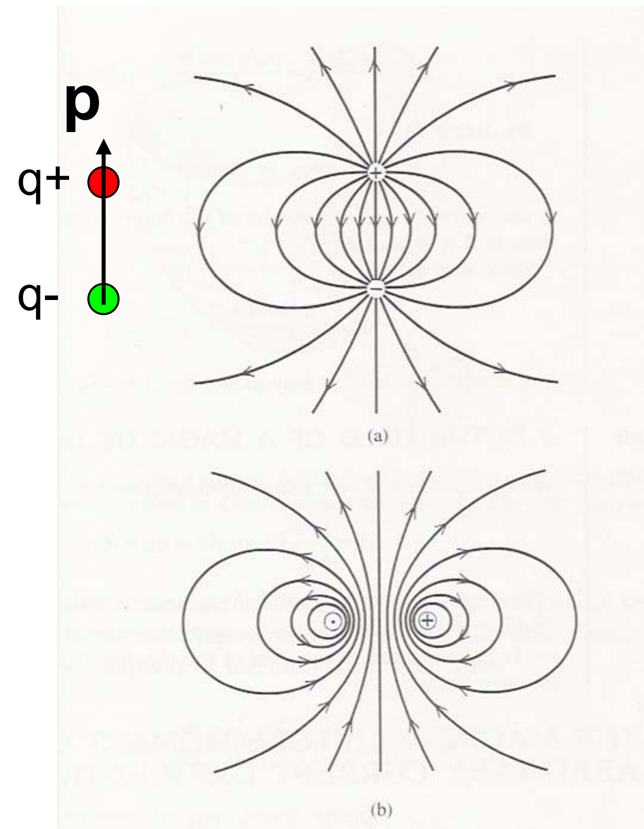
The off-axis field of circular loop is much more complex. For $z >> a$ (only) it is *identical* to that of the electric point dipole

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_o r^3}\left[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right]$$

$$\mathbf{B} = \frac{\mu_o m}{4\pi r^3}\left[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right]$$

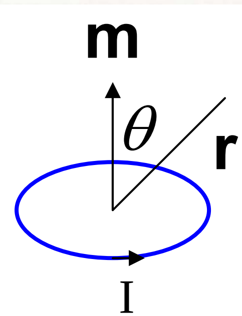
where $m = \pi a^2 I = \alpha I$ or $\mathbf{m} = \pi a^2 I\,\hat{z}$

a area enclosed by current loop




**m** = current times area *vs* **p** = charge times distance

**m** = magnetic dipole moment



# Differential form of Ampere's Law

Integral form of law: enclosed current is integral d**S** of current density **j**

$$\oint \mathbf{B}.d\ell = \mu_o I_{encl} = \mu_o \int_S \mathbf{j}.d\mathbf{S}$$

Apply Stokes' theorem

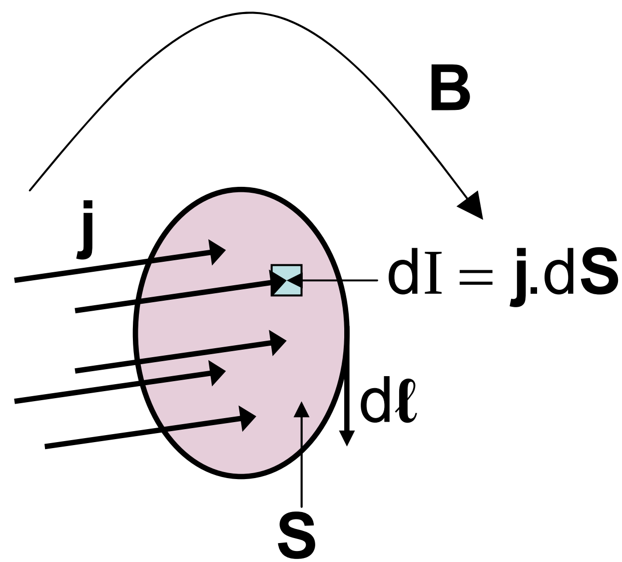
$$\oint \mathbf{B}.d\ell = \int_S (\nabla \times \mathbf{B}).d\mathbf{S} = \mu_o \int_S \mathbf{j}.d\mathbf{S}$$

$$\int_S (\nabla \times \mathbf{B} - \mu_o \mathbf{j}).d\mathbf{S} = 0$$

Integration surface is arbitrary

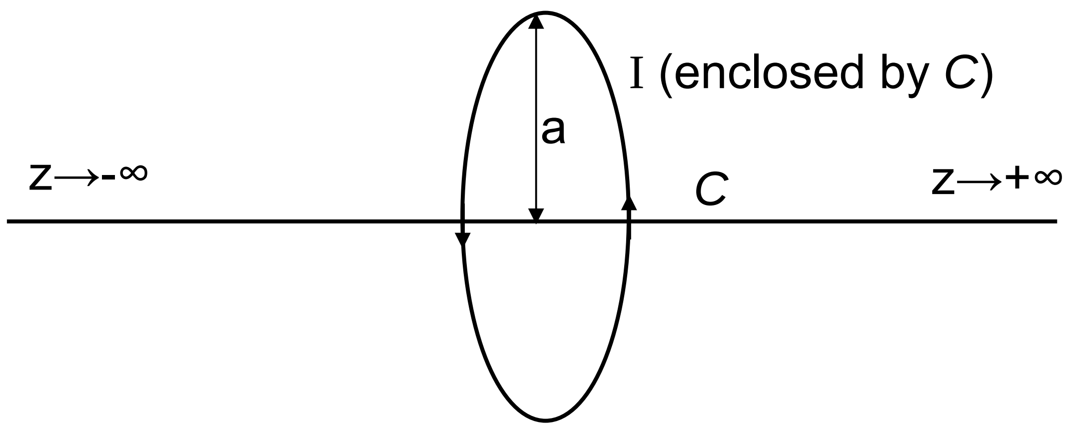
$$\nabla \times \mathbf{B} = \mu_o \mathbf{j}$$

Must be true point wise



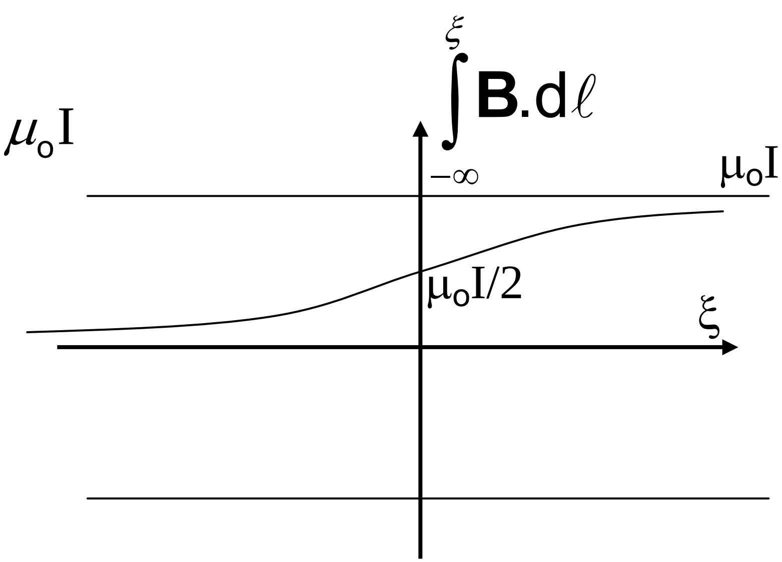
# **B**.d*ℓ* for current loop

- Consider line integral **B**.d*ℓ* from current loop of radius *a*
- Contour *C* is closed by large semi-circle, contributes zero to line integral




$$\oint_C \mathbf{B}.d\ell = \frac{\mu_o I}{2}\int_{-\infty}^{\infty}\frac{a^2 dz}{\left(a^2+z^2\right)^{3/2}} + 0\,(\text{semi}-\text{circle}) = \mu_o I$$

$$\int_{-\infty}^{\infty}\frac{a^2 dz}{\left(a^2+z^2\right)^{3/2}} = 2$$

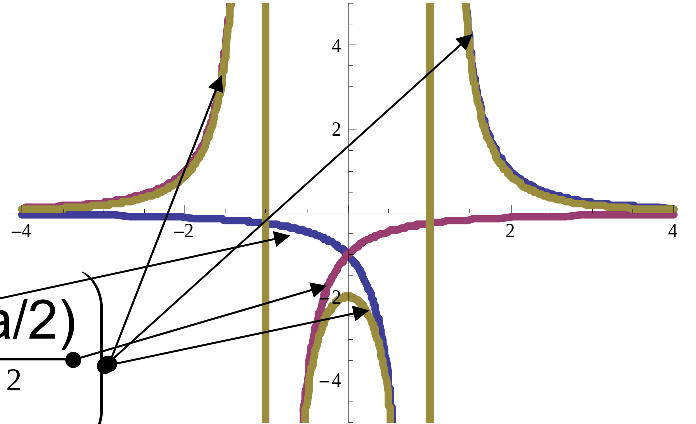


# $\mathbf{E}.d\ell$ for electric dipole

- Consider line integral $\mathbf{E}.d\ell$ for electric dipole with charges ±q at ±a/2
- Contour $C$ is closed by large semi-circle, contributes zero to line integral

$$\phi(z)=\frac{q}{4\pi\varepsilon_o}\left(\frac{1}{|z-a/2|}-\frac{1}{|z+a/2|}\right)$$

$$E_z(z)=-\frac{d\phi(z)}{dz}=\frac{q}{4\pi\varepsilon_o}\left(\frac{\mathrm{sign}(z-a/2)}{|z-a/2|^2}-\frac{\mathrm{sign}(z+a/2)}{|z+a/2|^2}\right)$$

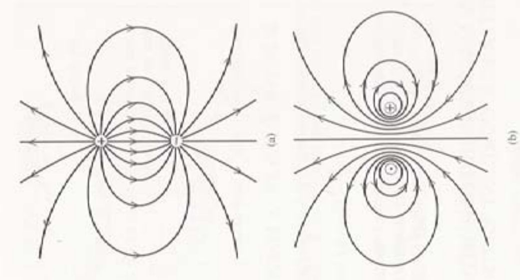



$$\int_{-\infty}^{\infty}E_z(z)d\ell+0\,(\mathrm{semi-circle})=\oint_C \mathbf{E}.d\ell=0$$

$\oint_C \mathbf{E}.d\ell$ vanishes for electrostatic field

$\oint_C \mathbf{B}.d\ell$ does not vanish when a current is enclosed by $C$

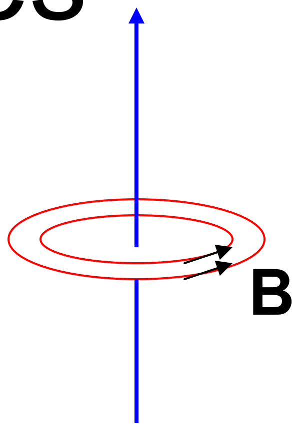
Electric Magnetic



Field reverses No reversal

# Ampere's Law examples

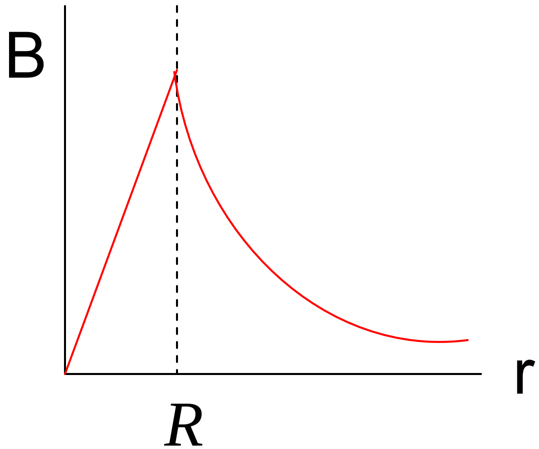
(1) Infinitely long, thin conductor




**B** is constant on circle of radius r

$$\oint \mathbf{B}.d\ell = \mu_o I_{encl} \Rightarrow B\,2\pi\,r = \mu_o I \Rightarrow B = \frac{\mu_o I}{2\pi\,r}$$

Exercise: Find radial profile of **B** *inside* conductor of radius R

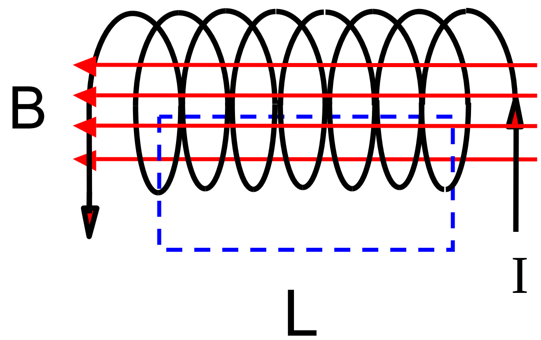



$$B_{r<R} = \frac{\mu_o I r}{2\pi R^2}$$

$$B_{r>R} = \frac{\mu_o I}{2\pi\,r}$$

# Ampere's Law examples

(2) Solenoid with N loops/metre




**B** constant and axial inside, zero outside
Rectangular path, axial length L

$$\oint \mathbf{B}.\mathrm{d}\ell = \mu_o I_{encl} \Rightarrow BL = \mu_o (NL) I \Rightarrow B = \mu_o N I$$

Exercise: Find **B** *inside* toroidal solenoid, i.e. one which forms a doughnut

*solenoid is to magnetostatics what capacitor is to electrostatics*