2.24. Subsequence

A sequence $\{y_n\}$ is called a subsequence of the sequence $\{x_n\}$ if there exists a sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 < n_3 < \ldots$ and $y_k = x_{n_k}$.

In other words, if we are given a sequence $\{x_n\}$ and a sequence $n_1 < n_2 < n_3 < \ldots$ of positive intergers, we select the terms of $\{x_n\}$ corresponding to the sequence $\{n_k\}$ and place them in the same order. This new obtained sequence is called a subsequence of $\{x_n\}$.

Method to construct a subsequence

Step I. Find a strictly monotonic increasing sequence of positive integers $n_1, n_2, n_3, \ldots$

i.e., $n_1 < n_2 < n_3 < \ldots$.

Step II. Images $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ of $n_1, n_2, n_3, \ldots$ under sequence $\{x_n\}$ are the elements of the subsequence $\left\{x_{n_k}\right\} = \{y_n\}$.

Examples.

(*i*) Let $n_k = 2k$, $k = 1, 2, 3, \ldots$

Now $\{n_k\} = \{2, 4, 6, \ldots\}$ is a strictly monotonic increasing sequence of positive integers.

$$\therefore \quad \left\{x_{n_k}\right\} = \{x_{2k}\} = \{x_2, x_4, x_6, \ldots\} \text{ is a subsequence of } \{x_n\}.$$

(*ii*) Let $n_k = 2k - 1$, $k = 1, 2, 3, \ldots$

Now $\{n_k\} = \{1, 3, 5, \ldots\}$ is strictly monotonic increasing sequence of positive integers.

$$\therefore \quad \left\{x_{n_k}\right\} = \{x_{2k-1}\} = \{x_1, x_3, x_5, \ldots\} \text{ is a subsequence of } \{x_n\}.$$

(*iii*) Let $n_k = k^2$, $k = 1, 2, 3, \ldots$

Now $\{n_k\} = \{1, 4, 9, \ldots\}$ is a strictly monotonic increasing sequence of positive intergers.

$$\therefore \quad \left\{x_{n_k}\right\} = \{x_{k^2}\} = \{x_1, x_4, x_9, \ldots\} \text{ is a subsequence of } \{x_n\}.$$

(iv) Let $n_k = k^3$, $k = 1, 2, 3, \ldots$

Now $\{n_k\} = \{1, 8, 27, \ldots\}$ is a strictly monotonic increasing sequence of positive integers.

$$\therefore \quad \left\{x_{n_k}\right\} = \left\{x_{k^3}\right\} = \{x_1, x_8, x_{27}, \ldots\} \text{ is a subsequence of } \{x_n\}.$$

Note. (1) Every sequence is a subsequence of itself.

(2) As $\{n_k\}$ is a strictly increasing sequence of positive integers, therefore the order in which the various terms of subsequence occur is the same as that in which they occur in the given sequence. Thus $\{8, 2, 4, 6, \ldots\}$ is not a subsequence of $\{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$.

(3) The interval between two consecutive terms of a subsequence is not always the same.

(4) If $x_m \in \{x_n\}$, then there exists an $n_i > m$ such that x_{n_i} belongs to the subsequence.

(5) Any subsequence of sequence is itself a sequence.

(6) A sequence has an infinite number of subsequences.

2.25. (i) If a sequence $\{x_n\}$ converges to l, then prove that every subsequence of $\{x_n\}$ also converges to l.

(ii) If a sequence $\{x_n\}$ diverges to $+\infty$, then prove that every subsequence of $\{x_n\}$ also diverges to $+\infty$.

(iii) If a sequence $\{x_n\}$ diverges to $-\infty$, then prove that every subsequence of $\{x_n\}$ also diverges to $-\infty$.

Proof. (i) Since $\{x_n\}$ converges to l

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$|x_n - l| < \varepsilon \quad \forall \quad n \geq m$$

If $n_p \geq m$ is a natural number, then for $k \geq p, n_k \geq n_p \geq m$

$$\therefore \quad |x_n - l| < \varepsilon \quad \forall \quad n_k \geq m$$

$\therefore$ subsequence $\left\{x_{n_k}\right\}$ also converges to l.

(ii) Since $\{x_n\}$ diverges to $+\infty$

$\therefore$ given $\Delta > 0$, however large, there exists a positive integer m such that

$$x_n > \Delta \quad \forall \quad n \geq m$$

If $n_p \geq m$ is a natural number, then for $k \geq p, n_k \geq n_p \geq m$

$\therefore$ $x_{n_k} > \Delta \quad \forall\ n_k \geq m$

$\Rightarrow$ subsequence $\left\{x_{n_k}\right\}$ diverges to $+\infty$.

(*iii*) Since $\{x_n\}$ diverges to $-\infty$

$\therefore$ given $\Delta > 0$, however large, there exists a positive integer m such that

$$x_n < -\Delta \quad \forall\ n \geq m$$

If $n_p \geq m$ is a natural number, then for $k \geq p, n_k \geq n_p \geq m$

$\therefore$ $x_{n_k} < -\Delta \quad \forall\ n_k \geq m$

$\Rightarrow$ subsequence $\left\{x_{n_k}\right\}$ diverges to $-\infty$.

Note. The converse of the above theorem is not true.

Examples. (*i*) Let $x_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, \ldots\}$

Two subsequences $\{-1, -1, -1, \ldots\}$ and $\{1, 1, 1, \ldots\}$ converges to -1 and 1 respectively. But $\{x_n\}$ does not converge.

(*ii*) Let $x_n = \begin{cases} n^2, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$

Now $\{x_{2n}\}$ diverges to $+\infty$ but $\{x_n\}$ does not diverge to $+\infty$.

(*iii*) Let $x_n = \begin{cases} -n^2, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$

Now $\{x_{2n-1}\}$ diverges to $-\infty$ but $\{x_n\}$ does not diverge to $-\infty$.

2.26. Peak Point of Sequence

(Pbi. U. 2010)

A natural number m is called a peak point of the sequence $\{x_n\}$ if $x_n < x_m \ \forall\ n > m$.

Examples. (*i*) Every natural number is a peak point of the sequence $\left\{\frac{1}{n}\right\}$. In fact every natural number is a peak point of strictly monotonic decreasing sequence.

(*ii*) Let $x_n = \begin{cases} \frac{1}{n}, & n = 1, 2, 3, \ldots, m \\ -1, & n > m \end{cases}$

$\therefore$ $\{x_n\}$ has exactly m peak points $1, 2, 3, \ldots, m$.

(iii) Let $x_n = n^2 \ \forall\, n \in \mathbb{N}$. Then $\{x_n\}$ has no peak point.

In fact a strictly increasing sequence has no peak point.

Note. A sequence may have no peak point, finite number of peak points or infinite number of peak points.

2.27. Prove that every sequence contains a monotone subsequence.

(G.N.D.U. 2017)

Proof. Three cases arise :

Case I. *The sequence $\{x_n\}$ has an infinite number of peak points.*

Let the peak points be $n_1, n_2, n_3, \ldots$, such that

$$n_1 < n_2 < n_3 < \ldots$$

$\therefore \quad \left\{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\right\}$ *i.e.,* $\left\{x_{n_k}\right\}$ is a subsequence of $\{x_n\}$

$\because \quad n_1$ is a peak point and $n_2 > n_1$

$\therefore \quad x_{n_2} < x_{n_1} \quad \Rightarrow \quad x_{n_1} > x_{n_2}$

Again n_2 is a peak point and $n_3 > n_2$

$\therefore \quad x_{n_3} < x_{n_2} \Rightarrow x_{n_2} > x_{n_3}$

$\therefore \quad x_{n_1} > x_{n_2} > x_{n_3}$

Proceeding in this way, we get,

$$x_{n_1} > x_{n_2} > x_{n_3} > \ldots$$

$\Rightarrow \quad \left\{x_{n_k}\right\}$ is a monotonic decreasing subsequence of $\{x_n\}$.

Case II. *The sequence $\{x_n\}$ has a finite number of peak points.*

Let $m_1, m_2, m_3, \ldots, m_p$ be the peak points of $\{x_n\}$.

Let n_1 be a natural number strictly greater than each of $m_1, m_2, \ldots, m_p$

$\because \quad n_1$ is not a peak point

$\therefore \quad$ there exists a natural number $n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$

Again n_2 is not a peak point of $\{x_n\}$

$\therefore \quad$ there exists a natural number $n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$.

Therefore we have $n_1 < n_2 < n_3$ such that $x_{n_1} \leq x_{n_2} \leq x_{n_3}$.

Proceeding in this way, we get a monotonic increasing subsequence $\left\{x_{n_k}\right\}$ of $\{x_n\}$.

Case III. *The sequence $\{x_n\}$ has no peak point.*

$\therefore$ 1 is not a peak point of $\{x_n\}$

$\therefore$ there exists a natural number $n_2 > 1 = n_1$ such that $x_{n_2} \geq x_{n_1}$

Again n_2 is not a peak point of $\{x_n\}$

$\therefore$ there exists a natural number $n_3 > n_2$ such $x_{n_3} \geq x_{n_2}$.

Therefore we have $n_1 < n_2 < n_3$ such that $x_{n_1} \leq x_{n_2} \leq x_{n_3}$.

Proceeding in this way, we get a monotonic increasing subsequence $\left\{x_{n_k}\right\}$ of $\{x_n\}$.

Hence every sequence contains a monotone subsequence.

Cor. Bolzano-Weierstrass Theorem

Prove that every bounded sequence has a convergent subsequence.

(Pbi. U. 2011)

Proof. Let $\{x_n\}$ be a bounded sequence.

$\because$ $\{x_n\}$ is a sequence, therefore $\{x_n\}$ has a monotone subsequence $\left\{x_{n_k}\right\}$

$\because$ $\{x_n\}$ is bounded, therefore $\left\{x_{n_k}\right\}$ is also bounded

[$\because$ every subsequence of a bounded sequence is bounded]

$\therefore$ $\left\{x_{n_k}\right\}$ is a bounded monotone sequence

$\Rightarrow$ $\left\{x_{n_k}\right\}$ is convergent

$\therefore$ $\{x_n\}$ has a convergent subsequence $\left\{x_{n_k}\right\}$

Hence the result.

2.28. Subsequential Limit or Cluster Point of a Sequence

A real number l is called a subsequential limit or cluster point of the sequence $\{x_n\}$ if there exists a sub-sequence of $\{x_n\}$ which converges to l.

Note. (1) If a sequence $\{x_n\}$ converges to l, then l is the only cluster point of $\{x_n\}$. This is so as every subsequence of $\{x_n\}$ converges to l.

(2) If a sequence has more than one cluster point, then it cannot be convergent.

(3) If a sequence $\{x_n\}$ diverges to $+\infty$, then $+\infty$ is the only cluster point of $\{x_n\}$.

(4) If a sequence $\{x_n\}$ diverges to $-\infty$, then $-\infty$ is the only cluster point of $\{x_n\}$.

Examples. (*i*) Consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n}$. Then the sequence $\{x_n\}$ converges to 0. Hence 0 is the only cluster point of $\{x_n\}$.

(*ii*) Consider the sequence $\{x_n\}$ where $x_n = (-1)^n$. The subsequence $\{-1, -1, -1, \ldots\}$ converges to 1 and subsequence $\{1, 1, 1, \ldots\}$ converges to 1. Therefore -1 and 1 are two cluster points of $\{x_n\}$.

(*iii*) Let $x_n = \begin{cases} n \ , \ n \text{ is odd} \\ -n \ , \ n \text{ is even} \end{cases}$

Subsequence $\{x_1, x_3, x_5, \ldots\}$ divergers to $+\infty$ and subsequence $\{x_2, x_4, x_6, \ldots\}$ diverges to $-\infty$. Thus the sequence $\{x_n\}$ has cluster points as $+\infty$ and $-\infty$.

2.29. Prove that a real number l is a limit point of a set A iff there exists a sequence of distinct points of A converging to l.

Proof. (*i*) Assume that $\{x_n\}$ is a sequence of distinct points of A converging to l.

$\therefore$ every neighbourhood of l contains infinitely many points of $\{x_n\}$ which are also points of A. Thus every nbd. of l contains infinitely many points of A, which in turn shows that l is the limit point of A.

(*ii*) Assume that l is a limit point of A.

$\therefore$ every nbd. of l contains infinitely many points of A.

$\therefore$ $\forall\, n \in \mathbf{N}, \mathrm{I}_n = \left(l - \frac{1}{n}, l + \frac{1}{n}\right)$ contains infinitely many points of A.

Choose $x_1 \in \mathrm{I}_1 \cap \mathrm{A}$ and the choose $x_2 \in \mathrm{I}_2 \cap \mathrm{A}$ such that $x_2 \neq x_1$.

Proceeding in this way, we choose $x_k \in \mathrm{I}_k \cap \mathrm{A}$ such that x_k is different from $x_1, x_2, \ldots, x_{k-1}$.

[This is possible as I_k contains infinitely many points of A]

$\therefore$ we get a sequence $\{x_n\}$ of distinct points of A such that $x_n \in \mathrm{I}_n$.

Let m be any fixed positive integer

$\therefore$ $\forall\, n \geq m, \frac{1}{n} \leq \frac{1}{m}$ and $-\frac{1}{n} \geq -\frac{1}{m}$

$\therefore$ $x + \frac{1}{n} \leq x + \frac{1}{m}$ and $x - \frac{1}{n} \geq x - \frac{1}{m}$

$\therefore$ $\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \subset \left(x - \frac{1}{m}, x + \frac{1}{m}\right)$

or $\mathrm{I}_n \subset \mathrm{I}_m \quad \forall\, n \geq m$

$\therefore$ $\forall\, n \geq m, x_n \in \mathrm{I}_n \Rightarrow x_n \in \mathrm{I}_m$

$\Rightarrow$ $x_n \in \left(x - \frac{1}{m}, x + \frac{1}{m}\right) \quad \forall\, n \geq m$

$\Rightarrow \quad |x_n - l| < \dfrac{1}{m} = \varepsilon \quad \forall \; n \geq m$

$\Rightarrow \quad \underset{n \to \infty}{\text{Lt}} \; x_n = l$

$\therefore$ sequence $\{x_n\}$ converges to l.

2.30. Prove that a real number l is a cluster point of real sequence $\{x_n\}$ iff given $\varepsilon > 0$, the interval $(l - \varepsilon, l + \varepsilon)$ contains infinitely many points of $\{x_n\}$.

Proof. (*i*) Assume that l is a cluster point of $\{x_n\}$

$\therefore$ there exists subsequence $\left\{x_{n_k}\right\}$ of $\{x_n\}$ which converges to l.

$\therefore$ given $\varepsilon > 0$, there exists a natural number m such that

$x_{n_k} \in (l - \varepsilon, l + \varepsilon) \quad \forall \; n_k \geq m.$

In particular, $x_n \in (l - \varepsilon, l + \varepsilon)$ for infinitely many n *i.e.,* the interval $(l - \varepsilon, l + \varepsilon)$ contains infinitely many terms $\{x_n\}$.

(*ii*) Assume that the interval $(l - \varepsilon, l + \varepsilon)$ contains infinitely many terms of $\{x_n\}$, where $\varepsilon > 0$.

$\therefore \quad x_n \in \left(l - \dfrac{1}{n}, l + \dfrac{1}{n}\right)$ for infinitely many n, where $\dfrac{1}{n} = \varepsilon$

In particular we can find $x_{n_1} \in (l - 1, l + 1)$.

Again $\left(l - \dfrac{1}{2}, l + \dfrac{1}{2}\right)$ contains x_n for infinitely many n, we can find $n_2 > n_1$ such that $x_{n_2} \in \left(l - \dfrac{1}{2}, l + \dfrac{1}{2}\right)$.

Proceeding in this way, we can find natural numbers

$n_1 < n_2 < n_3 < \ldots < n_k < \ldots$ such that

$x_{n_k} \in \left(l - \dfrac{1}{k}, l + \dfrac{1}{k}\right).$

i.e. $\left|x_{n_k} - l\right| < \dfrac{1}{k} \qquad \Rightarrow x_{n_k} \to l$

$\therefore$ l is a cluster point of $\{x_n\}$

2.31. Prove that a sequence $\{x_n\}$ converges to a real number l iff $\{x_n\}$ is bounded and l is the only cluster point of $\{x_n\}$

Proof. (*i*) Assume that $\{x_n\}$ converges to l

$\therefore$ $\{x_n\}$ is bounded and l is the only cluster point of $\{x_n\}$.

(*ii*) Assume that $\{x_n\}$ is bounded and l is the only cluster point of $\{x_n\}$.

If possible, suppose that $\{x_n\}$ does not converge to l.

$\therefore$ there exists an $\varepsilon > 0$, such that for $m \in \mathbf{N}$, there exists an $n \geq m$ such that

$x_n \notin (l - \varepsilon, l + \varepsilon)$.

In particular, there exists an integer n_1 such that $x_{n_1} \notin (l - \varepsilon, l + \varepsilon)$ on the basis of same argument, there exists an integer $n_2 > n_1$ such that $x_{n_2} \notin (l - \varepsilon, l + \varepsilon)$.

Proceeding in this way, we get a subsequence $\left\{x_{n_k}\right\}$ such that

$x_{n_k} \notin (l - \varepsilon, l + \varepsilon) \;\; \forall \;\; k$

Now $\{x_n\}$ is bounded $\Rightarrow$ $\left\{x_{n_k}\right\}$ is also bounded

$\therefore$ by Bolzano-Weierstrass Theorem, $\left\{x_{n_k}\right\}$ has a convergent subsequence converging to a real number $l' \neq l$ $\quad [\because \; x_{n_k} \notin (l - \varepsilon, l + \varepsilon)$ for any $k]$

$\therefore$ l' is a cluster point of $\{x_n\}$, which contradicts that l is the only cluster point of $\{x_n\}$.

$\therefore$ our supposition is wrong

$\therefore$ $\{x_n\}$ converges to l.

Note. Similarly we can prove that $\{x_n\}$ diverges to $+\infty$ or $-\infty$ iff $+\infty$ or $-\infty$ is the only cluster point of $\{x_n\}$.

2.32. (*i*) If a sequence $\{x_n\}$ converges to l, then its subsequences $\{x_{2n+1}\}$ and $\{x_{2n}\}$ also converge to l.

(*ii*) If the two sub-sequences $\{x_{2n+1}\}$ and $\{x_{2n}\}$ of a sequence $\{x_n\}$ converge to the same limit l, then $\{x_n\}$ also converges to l.

Proof : (*i*) Since $\{x_n\}$ converges to l

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$|x_n - l| < \varepsilon \quad \forall \; n \geq m$

Now $2n > m$ and $2n+1 > m$

$\therefore \quad |x_{2n} - l| < \varepsilon \quad \forall \quad n \geq m$

and $|x_{2n+1} - l| < \varepsilon \quad \forall \quad n \geq m$

$\therefore \quad x_{2n} \to l$ and $x_{2n+1} \to l$

$\therefore$ subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to l.

(*ii*) Now $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to l

$\therefore \quad x_{2n} \to l$ and $x_{2n+1} \to l$

$\therefore$ given $\varepsilon > 0$, $\exists$ natural numbers m_1 and m_2 s.t.

$$|x_{2n} - l| < \varepsilon \quad \forall \quad n \geq m_1 \qquad \ldots(1)$$

and $$|x_{2n+1} - l| < \varepsilon \quad \forall \quad n \geq m_2 \qquad \ldots(2)$$

Two cases arise :

Case I. *n is even*

Let $n = 2k$

$\therefore \quad |x_n - l| = |x_{2k} - l| < \varepsilon \quad \forall \quad k \geq m_1 \qquad [\because$ of (1)]

Now $n = 2k \quad \Rightarrow \quad n \geq 2m_1$

$\therefore \quad |x_n - l| < \varepsilon \quad \forall \quad n \geq 2m_1 \qquad \ldots(3)$

Case II. *n is odd*

Let $n = 2k+1$

$\therefore \quad |x_n - l| = |x_{2k+1} - l| < \varepsilon \quad \forall \quad k \geq m_2 \qquad [\because$ of (2)]

Now $n = 2k+1 \quad \Rightarrow \quad n \geq 2m_2 + 1$

$\therefore \quad |x_n - l| < \varepsilon \quad \forall \quad n \geq 2m_2 + 1 \qquad \ldots(4)$

Let $m = \text{maximum}(2m_1, 2m_2 + 1)$

$\therefore$ from (3) and (4), we get,

$$|x_n - l| < \varepsilon \;\forall\; n \geq m$$

$\Rightarrow \quad x_n \to l$ as $n \to \infty$

$\Rightarrow \quad \{x_n\}$ converges to l.

2.33. Cauchy Sequence

A sequence $\{x_n\}$ is said to be a Cauchy sequence if given $\varepsilon > 0$, however small, $\exists$ a positive integer k (depending upon ε) such that

$$|x_n - x_m| < \varepsilon \quad \forall \quad n, m \geq k.$$

(G.N.D.U. 2008, 2013; P.U. 2010)

Another Def. A sequence $\{x_n\}$ is said to be a Cauchy sequence if given $\varepsilon > 0$, however small, $\exists$ a positive integer m (depending upon ε) s.t.

$$|x_{n+p} - x_n| < \varepsilon \quad \forall \quad n \geq m \text{ and } p \in \mathbf{N}.$$

2.34. Prove that a Cauchy sequence is bounded. (Pbi. U. 2010)

Proof : Let $\{x_n\}$ be Cauchy sequence.

$\therefore$ given $\varepsilon > 0$, there exists a positive integer p such that

$$|x_n - x_m| < \varepsilon \quad \forall \; n, m \geq p \qquad ...(1)$$

In particular, $|x_n - x_p| < \varepsilon \quad \forall \; n \geq p$ ...(2)

Now $|x_n| = |(x_n - x_p) + x_p| \leq |x_n - x_p| + |x_p|$

$< \varepsilon + |x_p| \quad \forall \; n \geq p$ [$\because$ of (2)]

$\therefore \quad |x_n| < \varepsilon + |x_p| \quad \forall \; n \geq p$

Let $M = \max. \{|x_1|, |x_2|, ..., |x_{p-1}|, \in + |x_p|\}$

$\therefore \quad |x_n| \leq M \quad \forall \; n$

$\Rightarrow \quad \{x_n\}$ is bounded.

2.35. Prove that a convergent sequence is always a Cauchy sequence.

(H.P.U. 2005; P.U. 2007 ; G.N.D.U. 2008; Pbi. U. 2009)

Proof. Let the sequence $\{x_n\}$ converge to l

$\therefore$ given $\varepsilon > 0$, however small, $\exists \; k \in \mathbf{N}$ s.t.

$$|x_n - l| < \frac{\varepsilon}{2} \quad \forall \; n \geq k \qquad ...(1)$$

Let $m \geq k$ be a natural number.

$$\therefore \quad |x_m - l| < \frac{\varepsilon}{2} \quad \forall \; m \geq k \qquad ...(2)$$

Now $|x_n - x_m| = |(x_n - l) + (l - x_m)| \leq |x_n - l| + |l - x_m|$

$$= |x_n - l| + |x_m - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad [\because \text{ of } (1), (2)]$$

$\therefore \quad |x_n - x_m| < \varepsilon \quad \forall \; n, m \geq k$

$\{x_n\}$ is a Cauchy sequence.

2.36. Prove that a Cauchy sequence is always convergent.

(H.P.U. 2010, 2012; Pbi. U. 2011; G.N.D.U. 2012, 2013)

Proof. Let $\{x_n\}$ be a Cauchy sequence.

$\therefore$ given $\varepsilon > 0$, there exists a positive integer p such that

$$|x_n - x_m| < \varepsilon \quad \forall \; n, m \geq p \qquad ...(1)$$

In particular, $|x_n - x_p| < \varepsilon \quad \forall \; n \geq p$ ...(2)

Now $|x_n| = |(x_n - x_p) + x_p| \leq |x_n - x_p| + |x_p|$

$< \varepsilon + |x_p| \quad \forall \; n \geq p$ [$\because$ of (2)]

$\therefore \quad |x_n| < \varepsilon + |x_p| \quad \forall \; n \geq p$

Let $M = \max. \{ |x_1|, |x_2|, \ldots, |x_{p-1}|, \varepsilon + |x_p| \}$

$\Rightarrow \quad \{x_n\}$ is bounded.

$\therefore$ by Bolzano-Weierstrass Theorem, $\{x_n\}$ has a convergent subsequence $\left\{x_{n_k}\right\}$.

Let $\left\{x_{n_k}\right\}$ be convergent to l. We shall prove that $\{x_n\}$ also converges to l.

Since $x_{n_k} \to l$

$\therefore$ given $\varepsilon > 0$, $\exists$ a positive integer p s.t.

$$\left|x_{n_k} - l\right| < \varepsilon \quad \forall \; k \geq p \qquad \ldots(3)$$

$\therefore$ for $n \geq p$, $n_k \geq n_p \geq p$, from (1), we have,

$$\left|x_n - x_{n_k}\right| < \frac{\varepsilon}{2}. \qquad \ldots(4)$$

$\therefore \quad |x_n - l| = \left|\left(x_n - x_{n_k}\right) + \left(x_{n_k} - l\right)\right| \leq \left|x_n - x_{n_k}\right| + \left|x_{n_k} - l\right|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall \; n \geq p$ [$\because$ of (3), (4)]

$= \varepsilon \quad \forall \; n \geq p$

$\therefore \quad |x_n - l| < \varepsilon \quad \forall \; n \geq p$

$\Rightarrow \quad \{x_n\}$ is convergent.

2.37 Cauchy's General Principle of Convergence

Prove that a necessary and sufficient condition for the convergence of a sequence $\{x_n\}$ of real numbers is that it is a Cauchy sequence.

(G.N.D.U. 2010, 2013, 2015, 2016 ; H.P.U. 2013 ; P.U. 2013;)

Proof. Reproduce Art-35 and Art-36.

Note : In the system of rational numbers, every Cauchy sequence does not converge to a rational number.

Consider the sequence 1·4, 1·41, 1·414, 1·4142. It is a Cauchy sequence but does not converge to a rational number. It converges to $\sqrt{2}$.

2.38. Cantor's Intersection Theorem

Let $\{I_n\}$ where $I_n = [a_n, b_n]$ be a sequence of the closed intervals such that

(i) $I_{n+1} \subset I_n \ \forall \ n$

(ii) $l(I_n) = b_n - a_n \to 0$ as $n \to \infty$, then $\exists$ a unique point c such that $c \in I_n \ \forall \ n$, $l(I_n)$ denotes the length of the interval I_n. (G.N.D.U. 2011)

Proof : Since $I_{n+1} \subset I_n \ \forall \ n$

$\therefore \quad a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \ \forall \ n$

$\therefore \quad a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$ ...(1)

$b_1 \geq b_2 \geq b_3 \geq \ldots \geq b_n \geq b_{n+1} \geq \ldots$ ...(2)

$\therefore$ sequence $\{a_n\}$ is monotonically increasing and is bounded above as $a_n < b_1 \ \forall \ n$

$\therefore \quad \{a_n\}$ is convergent.

Also $\{b_n\}$ is monotonically decreasing and is bounded below as $b_n > a_1 \ \forall \ n$.

$\therefore \quad \{b_n\}$ is convergent

Let $a_n \to \alpha$, $b_n \to \beta$

$\because \quad b_n = (b_n - a_n) + a_n$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} b_n = \underset{n \to \infty}{\text{Lt}} (b_n - a_n) + \underset{n \to \infty}{\text{Lt}} a_n$$

$\Rightarrow \quad \beta = 0 + \alpha \quad \Rightarrow \quad \alpha = \beta = c$ (say)

Now c is l.u.b. of $\{a_n\}$ and g.l.b. of $\{b_n\}$

$\therefore \quad a_n \leq c \leq b_n \quad \forall \ n$

$\Rightarrow \quad c \in [a_n, b_n] \quad \forall \ n$

$\Rightarrow \quad c \in I_n \quad \forall \ n.$

To show that c is unique

If possible, suppose that there exists two real numbers c and c' which belong to $I_n \ \forall \ n$.

$\therefore \quad b_n - a_n \geq |c - c'| \ \forall \ n$

$\therefore \quad \{b_n - a_n\}$ cannot converge to zero which contradicts that $b_n - a_n \to 0$

$\therefore \quad$ our supposition is wrong

$\therefore \quad c$ is unique.

Note : This theorem is also known as '*Cantor's Theorem on Nested Intervals*' or '*Nested Interval Property*'.

2.39. Limit Superior and Limit Inferior of a Sequence

We know that a sequence of real numbers always contains a monotone subsequence and a monotone sequence converges or diverges to $+\infty$ or $-\infty$. Thus if E denotes the set of all the cluster points (*i.e.*, subsequential limits), including $+\infty$ and $-\infty$, of a sequence $\{x_n\}$ of real numbers, then E has at least one element *i.e.*, E is non-empty.

(*i*) The *l.u.b* of E in the extended real number system is called the *limit superior* or *upper limit* of $\{x_n\}$ and is denoted by $\overline{\text{Lt}}_{n\to\infty} x_n$ or $\text{Lt}_{n\to\infty} \sup x_n$ or $\overline{\text{Lt}}\, x_n$ or $\text{Lt} \sup x_n$.

(*ii*) The *g.l.b* of E in the extended real number system is called the *limit inferior* or *lower limit* of $\{x_n\}$ and is denoted by $\underline{\text{Lt}}_{n\to\infty} x_n$ or $\text{Lt}_{n\to\infty} \inf. x_n$ or $\underline{\text{Lt}}\, x_n$ or $\text{Lt} \inf. x_n$.

Examples. (*i*) Let $x_n = (-1)^n$

$\therefore$ $\{x_n\}$ has only two cluster points -1 and 1

$\therefore$ $E = \{-1, 1\}$

$\therefore$ $\overline{\text{Lt}}\, x_n = 1$, $\underline{\text{Lt}}\, x_n = -1$

(*ii*) Let $\{x_n\}$ converge to l. Then every subsequence of $\{x_n\}$ converges to l.

$\therefore$ $E = \{l\}$

$\therefore$ $\overline{\text{Lt}}\, x_n = l$, $\underline{\text{Lt}}\, x_n = l$

(*iii*) Let $x_n = \begin{cases} n & , n \text{ is odd} \\ -n & , n \text{ is even} \end{cases}$

$\therefore$ $E = \{-\infty, \infty\}$

$\therefore$ $\overline{\text{Lt}}\, x_n = \infty$, $\underline{\text{Lt}}\, x_n = -\infty$

Properties of Limit Superior

For a bounded sequence $\{x_n\}$ $\overline{\text{Lt}}\, x_n = u$ iff for every $\varepsilon > 0$,

(*i*) there exists a natural number m such that $x_n < u + \varepsilon \;\forall\; n \geq m$,

(*ii*) $x_n > u - \varepsilon$ for infinitely many values of n.

Properties of Limit Inferior

For a bounded sequence $\{x_n\}$, $\underline{\text{Lt}}\, x_n = l$ iff for every $\varepsilon > 0$

(*i*) there exists a natural number m such that $x_n > l - \varepsilon \;\forall\; n \geq m$,

(*ii*) $x_n < l + \varepsilon$ for infinitely many values of n.

ILLUSTRATIVE EXAMPLES

Example 1. Prove that the sequence $\{a_n\}$ where $a_n = 8 + \frac{1}{n^3}$ is a Cauchy sequence and find its limit. (P.U. 2009)

Sol. Here $a_n = 8 + \frac{1}{n^3}$, $\therefore\ a_m = 8 + \frac{1}{m^3}$

Without loss of generality, we take $n > m$.

Let $\varepsilon > 0$, however small. Then $|a_n - a_m| < \varepsilon$

$$\text{if } \left|\left(8+\frac{1}{n^3}\right)-\left(8+\frac{1}{m^3}\right)\right| < \varepsilon$$

i.e., if $\left|\frac{1}{n^3}-\frac{1}{m^3}\right| < \varepsilon$

i.e., if $\frac{1}{m^3}-\frac{1}{n^3} < \varepsilon$ $\quad\left[\because n > m \Rightarrow \frac{1}{n^3} < \frac{1}{m^3}\right]$

i.e., if $\frac{1}{m^3} < \frac{1}{n^3} + \varepsilon$

i.e., if $\frac{1}{m^3} < \varepsilon$

i.e., if $m^3 > \frac{1}{\varepsilon}$

i.e., if $m > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{3}}$

Let p be any positive integer just greater than $\left(\frac{1}{\varepsilon}\right)^{\frac{1}{3}}$

$\therefore$ $|a_n - a_m| < \varepsilon\ \forall\ n, m \geq p$

$\Rightarrow$ $\{a_n\}$ is a cauchy sequence

$\Rightarrow$ $\{a_n\}$ is convergent as every Cauchy sequence is convergent.

Now $\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\left(8+\frac{1}{n^3}\right) = \underset{n\to\infty}{\text{Lt}}\ 8 + \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{n}\right)^3 = 8+\left(\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\right)^3$

$$= 8 + 0\left[\because \left\{\frac{1}{n}\right\} \text{ is a null sequence}\right]$$

$$= 8$$

$\therefore$ sequence $\{a_n\}$ converges to 8.

Example 2. Prove that $\left\{\dfrac{n^3}{n^3+1}\right\}$ is a cauchy sequence.

(G.N.D.U. 2018)

Sol. Let $a_n = \dfrac{n^3}{n^3+1}$. So $a_m = \dfrac{m^3}{m^3+1}$.

Without loss of generality, we take $n > m$. Let $\varepsilon > 0$, however small. Then

$$|a_n - a_m| < \varepsilon$$

if $$\left|\frac{n^3}{n^3+1} - \frac{m^3}{m^3+1}\right| < \varepsilon$$

i.e. if $$\left|\left(1-\frac{1}{n^3+1}\right) - \left(1-\frac{1}{m^3+1}\right)\right| < \varepsilon$$

i.e. if $$\left|\frac{1}{m^3+1} - \frac{1}{n^3+1}\right| < \varepsilon$$

i.e. if $$\frac{1}{m^3+1} - \frac{1}{n^3+1} < \varepsilon \quad \left[\begin{array}{lll} \because & n > m & \Rightarrow n^3 > m^3 \\ \Rightarrow & n^3+1 > m^3+1 & \Rightarrow \dfrac{1}{m^3+1} > \dfrac{1}{n^3+1} \end{array}\right]$$

i.e. if $$\frac{1}{m^3+1} < \frac{1}{n^3+1} + \varepsilon$$

i.e. if $$\frac{1}{m^3+1} < \varepsilon$$

i.e. if $$m^3 + 1 > \frac{1}{\varepsilon}$$

i.e. if $$m > \left(\frac{1}{\varepsilon} - 1\right)^{\frac{1}{3}}$$

Let p be any positive integer just greater than $\left(\dfrac{1}{\varepsilon} - 1\right)^{\frac{1}{3}}$

$\therefore \quad |a_n - a_m| < \varepsilon \;\; \forall \; n, m \geq p$

$\Rightarrow \quad \{a_n\}$ is a cauchy sequence.

$\Rightarrow \quad \{a_n\}$ is convergent as every cauchy sequence is convergent.

Example 3. Prove directly that the following sequences are Cauchy sequences :

(i) $\left\{\frac{1}{n}\right\}$ (ii) $\left\{\frac{1}{n^2}\right\}$ (iii) $\left\{\frac{(-1)^n}{n}\right\}$

(iv) $\left\{\frac{n}{n+1}\right\}$ (v) $\left\{\frac{n+1}{n}\right\}$

Sol. (i) Here $a_n = \frac{1}{n}$ $\therefore$ $a_m = \frac{1}{m}$

Without loss of generality, we take $n > m$.

Let $\varepsilon > 0$, however small. Then $|a_n - a_m| < \varepsilon$

if $\left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon$

i.e., if $\frac{1}{m} - \frac{1}{n} < \varepsilon$ $\left[\because n > m \Rightarrow \frac{1}{m} > \frac{1}{n}\right]$

i.e., if $\frac{1}{m} < \frac{1}{n} + \varepsilon$

i.e., if $\frac{1}{m} < \varepsilon$

i.e., if $m > \frac{1}{\varepsilon}$

Let p be any positive integer just greater than $\frac{1}{\varepsilon}$

$\therefore$ $|a_n - a_m| < \varepsilon$ $\forall$ $n, m \geq p$

$\Rightarrow$ $\{a_n\}$ is a cauchy sequence.

(ii) Here $a_n = \frac{1}{n^2}$ $\therefore$ $a_m = \frac{1}{m^2}$

Without loss of generality, we take $n > m$.

Let $\varepsilon > 0$, however small. Then $|a_n - a_m| < \varepsilon$

if $\left|\frac{1}{n^2} - \frac{1}{m^2}\right| < \varepsilon$

i.e., if $\frac{1}{m^2} - \frac{1}{n^2} < \varepsilon$ $\left[\because n > m \Rightarrow \frac{1}{m^2} > \frac{1}{n^2}\right]$

i.e., if $\frac{1}{m^2} < \frac{1}{n^2} + \varepsilon$

i.e., if $\frac{1}{m^2} < \varepsilon$

i.e., if $m^2 > \frac{1}{\varepsilon}$

i.e., if $m > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}}$

Let p be any positive integer just greater than $\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}}$

$\therefore \quad |a_n - a_m| < \varepsilon \quad \forall \quad n, m \geq p$

$\Rightarrow \quad \{a_n\}$ is a Cauchy sequence.

(*iii*) Here $a_n = \frac{(-1)^n}{n}$, $\quad \therefore \; a_m = \frac{(-1)^m}{m}$

Without loss of generality, we take $n > m$.

$$|a_n - a_m| = \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \left|\frac{(-1)^n}{n}\right| + \left|\frac{(-1)^m}{m}\right|$$

$$= \frac{1}{n} + \frac{1}{m} < \frac{1}{m} + \frac{1}{m} \qquad \left[\because \; n > m \;\Rightarrow\; \frac{1}{n} < \frac{1}{m}\right]$$

$$= \frac{2}{m}$$

Let $\varepsilon > 0$, however small. Then $|a_n - a_m| < \varepsilon$

if $\quad \frac{2}{m} < \varepsilon$

i.e., if $\quad m > \frac{2}{\varepsilon}$

Let p be any positive integer just greater than $\frac{2}{\varepsilon}$.

$\therefore \quad |a_n - a_m| < \varepsilon \quad \forall \, n, m \geq p$

$\Rightarrow \quad \{a_n\}$ is a Cauchy sequence.

(*iv*) Here $a_n = \frac{n}{n+1}$, $a_m = \frac{m}{m+1}$.

Without loss of generality, we take $n > m$

$$|a_n - a_m| = \left|\frac{n}{n+1} - \frac{m}{m+1}\right| = \left|\left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{m+1}\right)\right| = \left|\frac{1}{m+1} - \frac{1}{n+1}\right|$$

$$= \frac{1}{m+1} - \frac{1}{n+1} \qquad \left[\because \; n > m \;\Rightarrow\; \frac{1}{m+1} > \frac{1}{n+1}\right]$$

Let $\varepsilon > 0$, however small. Then $|a_n - a_m| < \varepsilon$

if $$\frac{1}{m+1} - \frac{1}{n+1} < \varepsilon$$

i.e., if $$\frac{1}{m+1} < \frac{1}{n+1} + \varepsilon$$

i.e., if $$\frac{1}{m+1} < \varepsilon$$

i.e., if $$m + 1 > \frac{1}{\varepsilon}$$

i.e., if $$m > \frac{1}{\varepsilon} - 1$$

Let p be any positive integer just greater than $\frac{1}{\varepsilon} - 1$.

$\therefore \quad |a_n - a_m| < \varepsilon \quad \forall \ n, m \geq p$

$\Rightarrow \quad \{a_n\}$ is a Cauchy sequence.

(v) Here $a_n = \frac{n+1}{n}$, $\therefore \ a_m = \frac{m+1}{m}$

Without loss of generality, we take $n > m$

$$|a_n - a_m| = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{m}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right|$$

$$= \frac{1}{m} - \frac{1}{n} \qquad \left[\because \ n > m \Rightarrow \frac{1}{n} < \frac{1}{m}\right]$$

Let $\varepsilon > 0$, however small. Then

$$|a_n - a_m| < \varepsilon$$

if $$\frac{1}{m} - \frac{1}{n} < \varepsilon$$

i.e., if $$\frac{1}{m} < \frac{1}{n} + \varepsilon$$

i.e., if $$\frac{1}{m} < \varepsilon$$

i.e., if $$m > \frac{1}{\varepsilon}$$

Let p be any positive integer just greater than $\frac{1}{\varepsilon}$.

$\therefore \quad |a_n - a_m| < \varepsilon \quad \forall \ n, m \geq p$

$\Rightarrow \quad \{a_n\}$ is a Cauchy sequence.

Example 4. Prove that the following sequences are not Cauchy sequences :

(*i*) $\{(-1)^n\}$ (*ii*) $\{(-1)^n n\}$ (*iii*) $\{n^2\}$

Sol. (*i*) Here $x_n = (-1)^n$

$\therefore \quad x_{2n} = (-1)^{2n} = 1, x_{2n+1} = (-1)^{2n+1} = -1$

Let $\varepsilon = 1$

Now $|x_{2n+1} - x_{2n}| = |-1-1| = |-2| = 2 > 1 = \varepsilon \ \forall \ n$

$\therefore \quad \{x_n\}$ is not a Cauchy sequence

(*ii*) Here $x_n = (-1)^n n$

$\therefore \quad x_{2n} = (-1)^{2n}(2n) = 2n, x_{2n+1} = (-1)^{2n+1}(2n+1) = -(2n+1)$

Let $\varepsilon = 1$

Now $|x_{2n+1} - x_{2n}| = |-2n-1-2n| = |-(4n+1)| = 4n+1 > 1 = \varepsilon \ \forall \ n$

$\therefore \quad \{x_n\}$ is not a Cauchy sequence.

(*iii*) Here $x_n = n^2$, $\quad \therefore \ x_{n+1} = (n+1)^2 = n^2 + 2n + 1$

Let $\varepsilon = 1$

$$\therefore \quad |x_{n+1} - x_n| = |n^2 + 2n + 1 - n^2| = |2n+1| = 2n+1 > 1 = \varepsilon \ \forall \ n$$

$\therefore \quad \{x_n\}$ is not a Cauchy sequence.

Example 5. Show that the sequence $\{a_n\}$ where $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ does not converge, by showing that it is not a Cauchy sequence.

(H.P.U. 2011 ; G.N.D.U. 2016 ; P.U. 2018,)

Sol. $\quad a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

$\therefore \quad a_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}$

Without any loss of generality, we take $m > n$.

$$\therefore \quad |a_m - a_n| = \left|\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{m}\right|$$

$$\text{or} \quad |a_m - a_n| = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{m}$$

Take $m = 2n$

$$\therefore \quad |a_{2n} - a_n| = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

$$\geq \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n}$$

$$\left[\begin{array}{l} \therefore\ n+1 \leq n+n,\ n+2 \leq n+n, \ldots, n+n \leq n+n \\ \because\ \frac{1}{n+1} \geq \frac{1}{n+n}, \frac{1}{n+2} \geq \frac{1}{n+n}, \ldots, \frac{1}{n+n} \geq \frac{1}{n+n} \end{array}\right]$$

$$= \frac{n}{2n} = \frac{1}{2}\ \forall\, n$$

$$\therefore \quad |a_{2n} - a_n| \geq \frac{1}{2}\ \forall\, n$$

$\{a_n\}$ is not Cauchy sequence

$\therefore$ $\{a_n\}$ does not converge.

Example 6. Show that the sequence $\{a_n\}$ where $a_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$ is not convergent. Prove that $\{a_n\}$ diverges to ∞. (P.U. 2017)

Sol. $a_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$

$$\therefore \quad a_m = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2m-1}$$

Let $m > n$

$$\therefore \quad |a_m - a_n| = \left|\frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{2m-1}\right|$$

$$= \frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{2m-1}$$

Take $m = 2n+1$

$$\therefore \quad |a_{2n+1} - a_n| = \frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{4n+1} > \frac{1}{4n+1} + \frac{1}{4n+1} + \ldots + \frac{1}{4n+1}$$

$$= \frac{n+1}{4n+1} = \frac{1}{4}\left(1 + \frac{3}{4n+1}\right) > \frac{1}{4}\ \forall\, n$$

$\therefore$ $\{a_n\}$ is not a Cauchy sequence.

$\Rightarrow$ $\{a_n\}$ is not convergent

Also $a_{n+1} = 1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}+\frac{1}{2n+1}$

$\therefore \quad a_{n+1} - a_n = \frac{1}{2n+1} > 0 \quad \forall \ n$

$\Rightarrow \quad a_{n+1} > a_n \ \forall \ n$

$\Rightarrow$ $\{a_n\}$ is monotonically increasing.

Now $\{a_n\}$ is monototically increasing and not convergent.

$\therefore$ $\{a_n\}$ diverges to ∞.

Example 7. Show that the sequence $\{a_n\}$ where $a_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}$ does not converge, by showing that it is not a Cauchy sequence. Prove that $\{a_n\}$ diverges to ∞.

(H.P.U. 2006, 2009; G.N.D.U. 2006; P.U. 2013)

Sol. $a_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}$

$\therefore \quad a_m = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2m}$

Let $m > n$

$\therefore \quad |a_m - a_n| = \left|\frac{1}{2n+2}+\frac{1}{2n+4}+\ldots+\frac{1}{2m}\right| = \frac{1}{2n+2}+\frac{1}{2n+4}+\ldots+\frac{1}{2m}$

Take $m = 2n$

$\therefore \quad |a_{2n} - a_n| = \frac{1}{2n+2}+\frac{1}{2n+4}+\ldots+\frac{1}{4n}$

$= \frac{1}{2n+2}+\frac{1}{2n+4}+\ldots+\frac{1}{2n+2n} > \frac{1}{4n}+\frac{1}{4n}+\frac{1}{4n}+\ldots+\frac{1}{4n}$

$= \frac{n}{4n} = \frac{1}{4} \quad \forall \ n$

$\therefore$ $\{a_n\}$ is not a Cauchy sequence

$\Rightarrow$ $\{a_n\}$ is not convergent.

Also $a_{n+1} = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}+\frac{1}{2n+2}$

$\therefore \quad a_{n+1} - a_n = \frac{1}{2n+2} > 0 \ \forall \ n$

$\therefore \quad a_{n+1} > a_n \ \forall \ n$

$\Rightarrow$ $\{a_n\}$ is monotonically increasing.

Now $\{a_n\}$ is monotonically increasing but not convergent.

$\therefore$ $\{a_n\}$ diverges to ∞.

Example 8. Apply Cauchy's General Principle of convergence to show that $\{a_n\}$ where $a_n = 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}$ converges.

(H.P.U. 2012, 2015 ; G.N.D.U. 2012, 2013, 2018 ; P.U. 2010, 2013)

Sol. Here $a_n = 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}$

$$\therefore \quad a_m = 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{m^2}$$

Let $n > m$

$$\therefore \quad |a_n - a_m| = \left|\frac{1}{(m+1)^2}+\frac{1}{(m+2)^2}+\ldots+\frac{1}{n^2}\right| = \frac{1}{(m+1)^2}+\frac{1}{(m+2)^2}+\ldots+\frac{1}{n^2}$$

$$< \frac{1}{m(m+1)}+\frac{1}{(m+1)(m+2)}+\ldots+\frac{1}{(n-1)n}$$

$$= \frac{(m+1)-m}{m(m+1)}+\frac{(m+2)-(m+1)}{(m+1)(m+2)}+\ldots+\frac{n-(n-1)}{(n-1)\,n}$$

$$= \left(\frac{1}{m}-\frac{1}{m+1}\right)+\left(\frac{1}{m+1}-\frac{1}{m+2}\right)+\left(\frac{1}{m+2}-\frac{1}{m+3}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$= \frac{1}{m}-\frac{1}{n} < \frac{1}{m}$$

$$\therefore \quad |a_n - a_m| = \frac{1}{m} < \varepsilon \text{ if } \frac{1}{m} < \varepsilon \quad \textit{i.e.}, \text{ if } m > \frac{1}{\varepsilon}$$

Let p be a positive integer just greater than $\frac{1}{\varepsilon}$

$\therefore \quad |a_n - a_m| < \varepsilon \;\forall\; n, m \geq p$

$\therefore$ $\{a_n\}$ is a Cauchy sequence

$\therefore$ by Cauchy General Principle of convergence, $\{a_n\}$ is convergent.

Example 9. Let $\{u_n\}$ be a sequence of positive real numbers such that

$u_{n+1} = \frac{1}{2}(u_n + u_{n-1}) \;\forall\; n \geq 2$. Then prove that $\{u_n\}$ converges to $\frac{1}{3}(u_1 + 2u_2)$.

(G.N.D.U. 2011)

Sol. Here $u_{n+1} = \frac{1}{2}(u_n + u_{n-1})$

First of all, we will prove that $\{u_n\}$ is a Cauchy sequence.

$$|u_{n+1}-u_n| = \left|\frac{u_n+u_{n-1}}{2}-u_n\right| = \left|\frac{u_{n-1}-u_n}{2}\right|$$

$$\therefore \quad |u_{n+1}-u_n| = \frac{1}{2}\,|u_n-u_{n-1}|$$

Changing n to $n-1, n-2, \ldots, 2$, we get,

$$|u_n-u_{n-1}| = \frac{1}{2}\,|u_{n-1}-u_{n-2}|$$

$$|u_{n-1}-u_{n-2}| = \frac{1}{2}\,|u_{n-2}-u_{n-3}|$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$|u_3-u_2| = \frac{1}{2}\,|u_2-u_1|$$

Multiplying these equations, we get

$$|u_{n+1}-u_n| = \frac{1}{2^{n-1}}\,|u_2-u_1| \qquad \ldots(1)$$

$\therefore$ for $n \geq m$,

$$|u_n-u_m| = |(u_n-u_{n-1})+(u_{n-1}-u_{n-2})+\ldots+(u_{m+1}-u_m)|$$

$$\leq |u_n-u_{n-1}|+|u_{n-1}-u_{n-2}|+\ldots+|u_{m+1}-u_m|$$

$$= \left(\frac{1}{2^{n-2}}+\frac{1}{2^{n-3}}+\ldots+\frac{1}{2^{m-1}}\right)|u_2-u_1| \qquad [\because \text{ of } (1)]$$

$$< |u_2-u_1|\left(\frac{1}{2^{m-1}}+\frac{1}{2^m}+\frac{1}{2^{m+1}}+\ldots\infty\right)$$

$$= |u_2-u_1|\,\frac{\frac{1}{2^{m-1}}}{1-\frac{1}{2}} \qquad \left[\because S_\infty = \frac{a}{1-r}\right]$$

$$= |u_2-u_1|\cdot\frac{1}{2^{m-2}}$$

$$\therefore \quad |u_n-u_m| < \frac{1}{2^{m-2}}\,|u_2-u_1| \quad \forall\ n \geq m$$

Given $\varepsilon > 0$, we can choose a positive integer p such that

$$\frac{1}{2^{m-2}}\,|u_2-u_1| < \varepsilon \quad \forall\ n, m \geq p$$

$\therefore$ $\{u_n\}$ is a Cauchy sequence.

$\Rightarrow$ $\{u_n\}$ is convergent.

Let $\underset{n \to \infty}{\text{Lt}} u_n = l$

Also $u_{n+1} = \frac{1}{2} (u_n + u_{n-1})$

Putting $n = 2, 3, 4, \ldots, n-1$, we get,

$$u_3 = \frac{1}{2} (u_2 + u_1)$$

$$u_4 = \frac{1}{2} (u_3 + u_2)$$

$$u_5 = \frac{1}{2} (u_4 + u_3)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$u_{n-1} = \frac{1}{2} (u_{n-2} + u_{n-3})$$

$$u_n = \frac{1}{2} (u_{n-1} + u_{n-2})$$

Adding these equations, we get, $u_n + \frac{1}{2} u_{n-1} = \frac{1}{2} (u_1 + 2 u_2)$

Taking limits as $n \to \infty$, we get,

$$l + \frac{1}{2} l = \frac{1}{2} (u_1 + 2 u_2)$$

$$\Rightarrow \quad \frac{3}{2} l = \frac{1}{2} (u_1 + 2 u_2) \quad \Rightarrow \quad l = \frac{1}{3} (u_1 + 2 u_2)$$

$\therefore$ $\{u_n\}$ converges to $\frac{1}{3} (u_1 + 2 u_2)$

Another Form : Let $\{x_n\}$ be a sequence of positive real numbers such that

$x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$, $\forall\ n \geq 3$. Prove that $\{x_n\}$ is a cauchy sequence and converges to $\frac{1}{3}(x_1 + 2x_2)$. (P.U. 2008)

Example 10. If $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are both Cauchy sequences, then $\{x_n\}$ need not be a Cauchy sequence.

Sol. Consider the sequence $\{x_n\}$ where $x_n = (-1)^n$.

Now $x_{2n} = (-1)^{2n} = 1 \quad \forall \ n \in \mathbf{N}$

$\therefore$ $\{x_{2n}\}$ is a constant sequence

$\Rightarrow$ $\{x_{2n}\}$ is a convergent sequence

$\Rightarrow$ $\{x_{2n}\}$ is a cauchy sequence.

Also $x_{2n+1} = (-1)^{2n+1} = -1 \quad \forall \ n \in \mathbf{N}$.

$\therefore$ $\{x_{2n+1}\}$ is a constant sequence

$\Rightarrow$ $\{x_{2n+1}\}$ is a convergent sequence.

$\Rightarrow$ $\{x_{2n+1}\}$ is a Cauchy sequence.

But $\{x_n\}$ is not a Cauchy sequence (Prove it)

Example 11. If the sequences $\{x_n\}$ and $\{y_n\}$ are convergent, then show by Cauchy General Principle of convergence that

(*i*) $\{x_n + y_n\}$ (*ii*) $\{x_n y_n\}$ are also convergent. (P.U. 2016)

Sol. Let $\varepsilon > 0$ be given

$\because$ $\{x_n\}$ is convergent

$\therefore$ $\{x_n\}$ is a Cauchy sequence

$\therefore$ $\exists$ positive integer p_1 such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \forall \ n, m \geq p_1 \qquad ...(1)$$

Again as $\{y_n\}$ is convergent

$\therefore$ $\{y_n\}$ is a Cauchy sequence

$\therefore$ there exists positive integer p_2 such that

$$|y_n - y_m| < \frac{\varepsilon}{2} \quad \forall \ n, m \geq p_2 \qquad ...(2)$$

Let p = maximum (p_1, p_2)

$\therefore$ (1) and (2) hold $\forall \ n, m \geq p$.

(*i*) $|(x_n + y_n) - (x_m + y_m)| = |(x_n - x_m) + (y_n - y_m)|$

$$\leq |x_n - x_m| + |y_n - y_m|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ $|(x_n + y_n) - (x_m + y_m)| < \varepsilon \quad \forall \ n, m \geq p$

$\Rightarrow$ $\{x_n + y_n\}$ is a Cauchy sequence and hence convergent.

(ii) Since $\{x_n\}$ and $\{y_n\}$ are both convergent

$\therefore$ $\{x_n\}$ and $\{y_n\}$ are both bounded

$\therefore$ there exist positive real numbers a and b such that

$$|x_n| < a, |y_n| < b \ \forall n \qquad \text{...(3)}$$

Now $|x_n y_n - x_m y_m| = |x_n y_n - x_n y_m + x_n y_m - x_m y_m|$

$$= |x_n (y_n - y_m) + y_m (x_n - x_m)|$$

$$\leq |x_n (y_n - y_m)| + |y_m (x_n - x_m)|$$

$$= |x_n| \, |y_n - y_m| + |y_m| \, |x_n - x_m|$$

$$< a \frac{\varepsilon}{2} + b \frac{\varepsilon}{2} \qquad [\because \text{ of } (3)]$$

$$= (a + b) \cdot \frac{\varepsilon}{2} \quad \forall \ n, m \geq p$$

$\therefore$ $|x_n y_n - x_m y_m| < \left(\frac{a+b}{2}\right) \varepsilon \ \forall \, n, m \geq p$

$\Rightarrow$ $\{x_n y_n\}$ is Cauchy sequence and hence convergent.