(a) **Commutative law**

$A \cup B = B \cup A,\ A \cap B = B \cap A$

(b) **Associative law**

$A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$

(c) **Distributive law–**

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) **Complimentation Law :**

S is a set & U is the univssal ret then compliment of S is S' or S^c 'or $\overline{S}$ when

$S' = \{ x : x \in U \text{ and } x \notin S\}$

or $S' = U - S;\ (S')' = S$

(e) **De morgan's law**

$(A \cup B)' = A' \cap B';\ (A \cap B)' = A' \cup B'$

Random Experiment : A random experiment is an experiment that is repeated under the same conditions and for which we have knowledge of all possible outcomes in advance, but the outcome of which can not be predicted with certainty.

For example, the experiment of throwing a die is a random experiment. Here the possible out comes are one or two or three or four or five or six. But for a particular throwing outcome can not be predicted.

Sample Space : The set of all possible outcomes for a random experiment is called sample space or event space. For example, if the ramdom experiment be throwing a die, then the sample space S is denoted by $S = \{1, 2, 3, 4, 5, 6\}$

Events : Any subset of the sample space S is called an event. The elements of the set are event points If E be the random experiment of throwing a die then $A = \{2, 4, 6\}$ is an event associated with E written as $A \subseteq S$.

Impossible event : The null set is $\phi \subseteq S$. The event ϕ is called impossible event i.e. the event contains no sample point.

Ex : The event "face seven" in rolling a die is an impossible event.

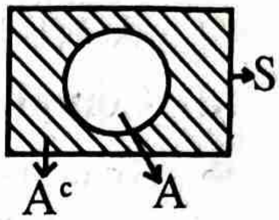
Certain or Sure event : Since $S \subseteq S$, therefore S itself is an event. The event S is called the certain event.

Ex : The event "any positive integer" in rolling a die is a certain event.

Complementary event : If E be a ramdom emperiment, S be the sample space, A be an event, then Event $S - A$ is called complementary event. It is dented by A' or A^c or $\overline{A}$

For example, if $S = \{1, 2, 3, 4, 5, 6\}$,

$A = \{2, 4, 6\}$, then $A^c = \{1, 3, 5\}$



Note : $S^c = \phi$, $\phi^c = S$, $(A^c)^c = A$,

$(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

Where $A \cup B$ denotes atleast one of A and B, $A \cap B$ denotes occurrence of both A and B.

Exhaustive events : The events A_1, A_2, A_3,....are said to be exhaustive if $A_1 \cup A_2 \cup A_3 \cup$............ = S.

For example, in the experiment rolling a die, the events

$A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$, $A_3 = \{4, 5, 6\}$

are exhaustive as $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6\} = S$

Mutually exclusive events : Two events A and B are said to be mutually exclusive if $A \cap B = \phi$.

For example, E = Rolling a die, $S = \{1, 2, 3, 4, 5, 6\}$. The event $A = \{1, 3, 5\}$ $B = \{2, 4, 6\}$ are mutually exclusive or disjoint as $A \cap B = \phi$.

Pairwise disjoint events : A collection $\{A_1, A_2, A_3, \ldots\ldots\ldots\}$ of events are said to pairwise disjoint if $A_i \cap A_j = \phi$. for $i \neq j$: i, j = 1,2,3,.......

Equally likely sample points : Let S be a finite sample space. Then its sample points are said to be equally likely if no one outcome is more likely to occur than the other.

For example, In a random toss of an unbiased or uniform coin, head and tail are equally likely events.

1.4 Probability of an event (classical definition of Probability) : Let E be the random experiment such that the sample space S contains a finite number n(S) of sample points, all of which are equally likely. Then probability of an event A, denoted by P(A), is defined by

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{favourable number cases}}{\text{total number of cases}}$$

For Example : A perfect die is rolled once (i) If A denotes the event of occurrence of odd face, then

A = {1, 3, 5}. Here S = {1, 2, 3, 4, 5, 6}

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

(ii) If A denotes the event of occurence of "multiple of three", then A = {3, 6}

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

1.4.1 Limitations of classical definition : This definition of classical probability breaks down in following cases.

(i) If the out comes of the random experiment are not equally likely or equally probable.

(ii) If the number of outcomes of the random experiment is infinite or unknown.

1.5 Frequency definition or statistical definition of Probability:

Let a random experiment E be repeated N times under same conditions, in which an event A is found to occur N(A) times, N(A) is called frequency of A; $\frac{N(A)}{N}$ is called relative frequency or frequency ratio of A. Then probability of A, denoted P(A) is defined b as $P(A) = \lim_{N \to \infty} \frac{N(A)}{N}$, if the limit exists.

1.5.1 Some deductions :

(i) For any event A, assoçiatated with random enperiment E, $0 \le P(A) \le 1$

Proof : Let E be random experiment which is repeated N times under identical condition, in which A is found to occur N(A) times

Therefore $0 \le N(A) \le N$

$$\text{or,} \quad 0 \le \frac{N(A)}{N} \le 1$$

Taking limit N tending to ∞ we get

$$0 \le \underset{N \to \infty}{Lt} \frac{N(A)}{N} \le 1$$

$$\text{or,} \quad 0 \le P(A) \le 1$$

(ii) Probability of certain event is equal to 1 i.e. if S is the sample space then P(S) = 1

Proof Here N(S) = N

$$\text{or,} \quad \frac{N(S)}{N} = 1$$

Taking N tending to infinity we get

$$\underset{N \to \infty}{Lt} \frac{N(S)}{N} = 1$$

$$\text{or,} \quad P(S) = 1$$

(iii) Probability of an impossible event is zero. i.e. P (ϕ) = 0 where ϕ is impossible event

Proof : Since N (ϕ) = 0

$$\therefore \quad \frac{N(\phi)}{N} = 0$$

$$\text{or,} \quad \underset{N \to \infty}{Lt} \frac{N(\phi)}{N} = 0$$

$$\text{or,} \quad P(\phi) = 0$$

(iv) If A and B are mutually exclusive events, then P(A ∪ B) = P(A) + P(B)

Proof : Let a random experiment E repeated N times, in which A occur N(A) times and B occurs N(B) times.

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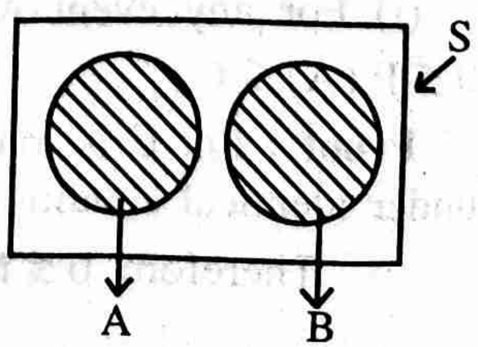
Since $A \cap B = \phi$

$$\therefore N(A \cup B) = N(A) + N(B)$$

$$\text{or,} \quad \frac{N(A \cup B)}{N} = \frac{N(A)}{N} + \frac{N(B)}{N}$$



Taking limit $N \to \infty$ we get

$$\operatorname*{Lt}_{N\to\infty} \frac{N(A\cup B)}{N} = \operatorname*{Lt}_{N\to\infty} \frac{N(A)}{N} + \operatorname*{Lt}_{N\to\infty} \frac{N(B)}{N}$$

or, $P(A \cup B) = P(A) + P(B)$

(v) If A_1, A_2, ... An are mutually exclusive then

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$$

Proof : This is an extension to any no of events by induction law.

1.6. If A and B are any two events, then

$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

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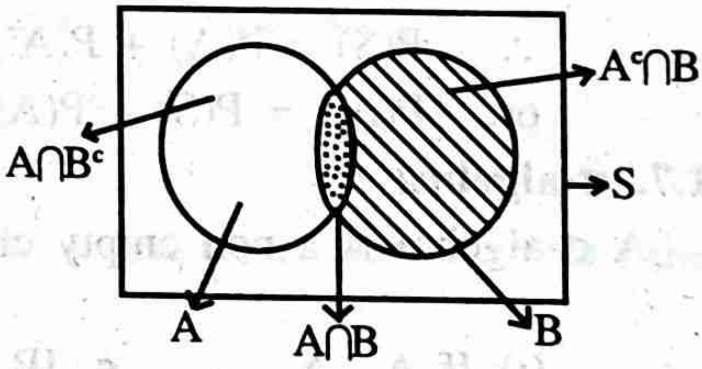
Proof : We have $A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$

The events $A \cap B^c$, $A \cap B$, $A^c \cap B$ are pairwise mutually exclusive.



$\therefore \quad P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$ .... (i)

Also, we can write $A = (A \cap B^c) \cup (A \cap B)$

$\therefore \quad P(A) = P(A \cap B^c) + P(A \cap B)$

or, $\quad P(A \cap B^c) = P(A) - P(A \cap B)$ .... (ii)

Similarly, $B = (A \cap B) \cup (A^c \cap B)$

$$P(B) = P(A \cap B) + P(A^c \cap B)$$

or, $\quad P(A^c \cap B) = P(B) - P(A \cap B)$ .... (iii)

From (i), using (ii) and (iii) we get

$$P(A \cup B) = P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B)$$

1.6.1 Extension for three events :

It A_1, A_2, A_3 are any three events then

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$$

Proof : $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2 \cup A_3) - P\{(A_1 \cap (A_2 \cup A_3)\}$

$= P(A_1) + P(A_2) + P(A_3) - P(A_2 \cap A_3) - \{P(A_1 \cap A_2) + P$
$P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3\}$

$[\because P\{A_1 \cap (A_2 \cup A_3)\} = P\{(A_1 \cap A_2) \cup (A_1 \cap A_3)\}$

and $(A_1 \cap A_2) \cap (A_1 \cap A_3) = A_1 \cap A_2 \cap A_3]$

$= P(A_1) + P(A_2) + P(A_3) - P(A_2 \cap A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$

1.6.2. Deductions :

For any event A. $P(A^c) = 1 - P(A)$

We have $S = A \cup A^c$ and $A \cap A^c = \phi$

$\therefore \quad P(S) = P(A) + P(A^c)$

or, $\quad P(A^c) = P(S) - P(A) = 1 - P(A)$ $[\therefore P(S) = 1]$

1.7. σ-algebra :

A σ-algebra is a non empty class IB of subsets of S, Such that

(i) If $A_1, A_2, \ldots\ldots\ldots \in$ IB, then $P(\bigcup_{i=1}^{\infty} A_i) \in IB$

(ii) If $A \in IB$, then $A^c \in IB$

σ-algebra is also called Boolean σ-algebra

Some Properties of σ-algebra :

(i) If $A_1, A_2, \ldots\ldots\ldots$ IB, then $\bigcap_{i=1}^{\infty} A_i \in IB$

Proof : $A_1, A_2, \ldots\ldots \in$ IB imples that
$A^c_1, A^c_2, \ldots\ldots \in$ IB

Again $\quad A^c_1, A^c_2, \ldots\ldots \in IB \quad \Rightarrow \bigcup_{i=1}^{\infty} A^c_i \in IB$

$\Rightarrow (\bigcap_{i=1}^{\infty} A_i)^c \in IB$ [By De Morgan's law]

$$\Rightarrow \left((\bigcap_{i=1}^{\infty} A_i)^c \right)^c \in IB$$

$$\Rightarrow \bigcap_{i=1}^{\infty} A_i \in IB$$

(ii) $S \in IB$

(iii) $\phi \in IB$

(iv) If $A \in IB$, $B \in IB$, than $A - B \in IB$

Proof : $A - B = A \cap B^c \in IB$ as $A \in IB$, $B^c \in IB$.

1.7.1 Axiomatic definition of Probability :

Let E be a random experiment and S be the sample space associated with E. Also let IB be a σ-algebra of subsets of S. Define a function or mapping $P : IB \rightarrow IR$, the set of real numbers. The function P is called probability measure or probability function if it satisfies the following axioms or conditions.

(i) $P(A) \geq 0$ for all $A \in IB$

(ii) $P(S) = 1$

(iii) If $A_1, A_2, \ldots\ldots\ldots$ are mutually exclusive events then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

P(A) is çalled Probability of an event A.

Ex 3. : What is the probability that a leap year selected at random will contain 53 wednesdays? W.B.U. Tech. 2002, 2007

Soln : A leap year contains 366 days. 366 days = 52 full weeks + two days (i) Monday and Tuesday (ii) Tuesday and Wednesday (iii) Wednesday and Thrusday (iv) Thrusday and Friday (v) Friday and Saturday (vi) Saturday and Sunday (vii) Sunday and Monday.

Therefore a leapyear will contain 53 Wednesday if one of the extra two days are Wednesdays.

∴ the required probability

$$= \frac{\text{Favourable no of cases}}{\text{Total no of cases}}$$

$$= \frac{2}{7}$$

Ex 4. : If A and B are two events such that $P(A) = P(B) = 1$, then show that $P(A \cup B) = 1$ and $P(A \cap B) = 1$

Soln : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 1 + 1 - P(A \cap B)$$

$$= 2 - P(A \cap B) \quad \text{.............(i)}$$

Now $P(A \cap B) \leq 1$

$\Rightarrow -P(A \cap B) \geq -1$

$\Rightarrow 2 - P(A \cap B) \geq 2 - 1 = 1$

$\Rightarrow P(A \cup B) \geq 1$

But $P(A \cup B) \leq 1$

$\therefore \quad P(A \cup B) = 1$

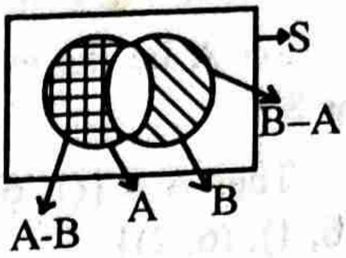
$P(A \cup B) = 2 - P(A \cap B)$

$\Rightarrow P(A \cap B) = 2 - P(A \cup B)$

$$= 2 - 1 = 1$$

Ex 5. : Show that the probability of occurence of only one of the events A and B is $P(A) + P(B) - 2P(A \cap B)$

Sol : The event that the occurence of only one of the events A and B is



$$(A - B) \cup (B - A)$$

$$= (A \cap B^c) \cup (B \cap A^c)$$

Now $P\{(A \cap B^c) \cup (B \cap A^c)\}$

$$= P(A \cap B^c) + P(B \cap A^c) \text{ .... (i)}$$

[$\because$ $(A \cap B^c)$ and $(B \cap A^c)$ are mutually exclusive

Now $A = (A \cap B^c) \cup (A \cap B)$

$\therefore \quad P(A) = P(A \cap B^c) + P(A \cap B)$

[$\because$ $(A \cap B^c)$ and $(A \cap B)$ are mutually exclusive events]

$\therefore \quad P(A \cap B^c) = P(A) - P(A \cap B)$

similarly $P(A^c \cap B) = P(B) - P(A \cap B)$

putting these values in (1) we get

$P\{(A \cap B^c) \cup (B \cap A^c)\} = P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= P(A) + P(B) - 2P(A \cap B)$ Proved.

Ex 6. : A bag contain 9 white and 4 black balls. If 6 balls are drawn at random, what is the probability that 3 are white and 3 are black.

Sol : 6 balls can be chosen from 13 balls in ${}^{13}c_6$ ways.

$\therefore$ Total number of cases = ${}^{13}c_6$

3 white balls can be chosen from 9 white balls in ${}^{9}c_3$ ways.

3 balcks balls can be chosen from 4 blacks balls in ${}^{4}c_3$ ways.

$\therefore$ favourable number of cases = ${}^{9}c_3 \times {}^{4}c_3$

$\therefore$ the required probability $= \dfrac{{}^{9}c_3 \times {}^{4}c_3}{{}^{13}c_6}$

$$= \frac{28}{143}$$

Ex 7. : When two dice are thrown, find the probability that the sum of the points on the dice is 7 or 8.

Ans. Here sample space is

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 4), (6, 5), (6, 6)}

Ex 12. : The number 1, 2, ...,n are arranged in random order. What is the probability that the numbers 1 and 2 are always together?

Soln : n numbers can be arranged in $\lfloor n$ ways.

$\therefore$ total number of cases $\lfloor n$.

Total number of cases where 1 and 2 are always together is $\lfloor 2 \times \lfloor n-1$

$$\therefore \text{ the required probability} = \frac{\lfloor 2 \times \lfloor n-1}{\lfloor n}$$

$$= \frac{2}{n}$$

Ex 13. : The integers x andy are choosen at random with replacement from the set of natural numbers {1, 2, ..., 9}. Find the probability that $|x^2 - y^2|$ is divisible by 2.

Soln : Two numbers x and y can be chooscn at random with replacement from 9 numbers in 9^2 ways.

$\therefore$ total numbner of cases = 9^2

$|x^2 - y^2|$ will be divisible by 2 if x and y are both even or x and y are both odd.

Let A be the event that "x and y are both even"

and B be the event that "x and y are both odd"

NowA contains 4^2 event points

B contains 5^2 event points

$\therefore$ $P(A \cup B) = P(A) + P(B)$ $[\because A \cap B = \phi]$

Now $P(A) = \frac{4^2}{9^2}$, $P(B) = \frac{5^2}{9^2}$

$$P(A \cup B) = \frac{16}{81} + \frac{25}{81}$$

$$= \frac{41}{81}$$